COMPARATIVE ANALYSIS OF SOME METHODS OF LYAAPUNOV CONSTRUCTIONS FOR THE CUBIC DUFFING OSCILLATOR-THE HARD SPRING MODEL

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Abstract.
In this paper, three different methods of construction of Lyapunov functions for Duffing-type equation were adopted and compared. Under appropriate consideration, similar results were obtained using different techniques for the hard spring system.

Keywords: Lyapunov Construction, Duffing Oscillator, The hard Spring Model

2010 Mathematics Subject Classification: 34B15, 34C15, 34C25, 34K13

1.0Introduction

The Lyapunov theorem on stability and asymptotic stability has been employed by many authors to study existence of a continuously differentiable Lyapunov function for non-linear differential equations of various types. Massera [1], Nijimeijer and Berghius [2], Macklin [3] andKrasovskii [4] gave necessary and sufficient conditions for the existence of a continuously differentiable Lyapunov function in some neighborhood of asymptotically stable unperturbed trajectory. [5 - 7] used Lyapunov theorem to show the existence of a suitable Lyapunov function for non-linear differential equations. The Duffing equation (oscillator):

$$\ddot{x} + c\dot{x} + a(x) + bx^3 = h(t) \qquad (1.1)$$

where *a,b,c* are real constants and *h* is continuous, has been widely used in physics, economics, engineering, and many other physical phenomena. Given its characteristic of oscillation and chaotic nature, many scientists are inspired by this nonlinear differential equation given its nature to replicate similar dynamics in our natural world. This equation together with Van der Pol's equation have become one of the most common examples of nonlinear oscillation in textbooks and research articles. See for instance [8 - 10] and the references therein. Due to the importance of the Duffing equation in real world problems, the study of convergence of solution of the equation using Lyapunov function has continued to attract the attention of many researchers. [11 - 14] have proposed independently the convergence of solution of Duffing equation of the general form:

$\ddot{x}+ f(x)\dot{x} + g(x)= p(t)$ (1.2)

wherep is continuous and 2π-periodic in $t \in R$, $f(x)$ is the stiffness term and

$g(x)$ is the nonlinear term

Motivated by the above results and ongoing research in this direction the

purpose of this paper is to consider the construction of Lyapunov functions using

different methods for the Duffing equation of the form:

$\ddot{x} + c\dot{x} + ax + bx^2+2x^3= h(t)$ (1.3)

where *a,b,c* are real constant and $h : [0,2\pi] \to R^n$ is continuous. In equation (1.3), a is the stiffness constant, c is the coefficient of viscous damping and $bx^2 + 2x^3$ represents the nonlinearity in the restoring force acting like a hard spring

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2.0 Preliminaries

Definition 2.1. (Properties of Lyapunov Function)The Lyapunov function has the following properties:

(i) Continuity: $V(t,X)$is continuous and single valued under the condition $t \geq T$ and $|x_i| < H \; and$

$$V(t,0) \equiv 0$$

(ii) $V\ (t,X)$ is positive definite

$(iii)\dot{V} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 + \cdots + \frac{\partial V}{\partial x_n}\dot{x}_n + \frac{\partial V}{\partial t}$, representing the total derivative with respect to t is negative definite.

Definition 2.2.(A Complete Lyapunov function)A Lyapunov function V defined as $V: I \times \mathbb{R}^n \to \mathbb{R}$is said to be COMPLETE iffor$X \in \mathbb{R}^n$,

$$(\boldsymbol{i})\ V(t,X) \geq 0$$

$$(ii)\ V(t,X) = 0 \; if \; and \; only \; if \; X = 0 \; and$$

$$(iii)\dot{V}(t,X) \leq -c|X| \; where \; c \; is \; any \; positive \; constant \; and \; |X| given \; by$$

$$|X| = \left(\sum_{i=1}^{n}(x_i^2)\right)^{\frac{1}{2}} \to \infty$$

It is INCOMPLETE if (iii) is not satisfied.

Definition 2.3.(Stability)Let $B(\bar{x},\varepsilon)$ denote the open ball centered at $\bar{x}$ of radius ε, that is the set $\{x \in \mathbb{R}^n : \| x - \bar{x} \| < \varepsilon\}$; $\bar{B}(\bar{x},\varepsilon)$ will denote a closed ball or the set $\{x \in \mathbb{R}^n : \| x - \bar{x} \| \leq \varepsilon\}$;

$\bar{B}(\bar{x},\varepsilon)$ *and* $S[\bar{x}, \varepsilon]$ denote the sphere or the set $\{x \in \mathbb{R}^n : \| x - \bar{x} \| = \varepsilon\}$.An equilibrium point x_e of a nonlinear system is said to be stable if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$\bar{x} \in B(x_\varepsilon, \delta) \Longrightarrow \varphi(t, 0, \bar{x}) \in B(x_\varepsilon, \varepsilon) \quad for\ all\ t \geq 0$.The Lyapunov stability of x_e assumes a "simultaneous continuity", more precisely the equicontinuity at x_e of all the functions in the $\{\phi_t : \bar{x} \to \varphi(t, 0, \bar{x})\ for\ t \geq 0$

Definition 2.4.(Asymptotic Stability)The equilibrium point x_e is said to be asymptotically stable, if for all $\varepsilon > 0$, the exists a $\delta > 0$ such that,

(i) $\varphi(t; 0, \bar{x}) \in B(x_e, \varepsilon)$ for all $t \geq 0$

(ii) $\lim_{t\to\infty} \varphi(t; 0, \bar{x}) = x_e$

Definition 2.5. (Lyapunov Functions)A continuously differentiable function $V: \mathbb{R}^n \to \mathbb{R}$ is called positive definite if $V(0) = 0$ and there exists an open ball $B = B(0,\varepsilon)$ such that

(i) $V(x) > 0$ for $x \in B$.

(ii)The function V is positive if there exists a $B\ such\ that\ V(0) = 0\ and\ V(x) \geq 0$ for all $x \in B$

(iii)Analogously V is called negative definite or negative semi-definite if

1. $V(0) = 0$and respectively

2. $V(x) < 0$ for all $x \in B, x \neq 0$ or$V(x) \leq 0$ for all $x \in B$.

Theorem 2.6. Consider the autonomous differential equation

$$\dot{x} = f(x) \qquad (2.1)$$

Suppose there exists a function $V: R^n \to R$ which is continuously differentiable and satisfies the following conditions

(i) $V(x)$ is positive definite i.e. $||x|| \leq V(x)$

(ii)The time derivative $\dot{V}$ of $V(x)$ along the solution path of equation (2.1) is negative semi-definite i.e.

$\dot{V} \leq 0$. Then the trivial solution $x = 0$ of equation (2.1) is locally stable (stable in the sense of Lyapunov)

Proof. Since $V(x)$ is positive definite then $V(0) = 0$ and $V(x) > 0 \ \Rightarrow \parallel V(x) \parallel = V(x)$ and

$$\parallel x \parallel \leq V(x) \qquad (2.2)$$

$V(x)$is continuously differentiable implies $V(x)$ is continuous and so continuous at the origin. So that given any $\varepsilon > 0$ $there\ exists$ $\delta > 0$ $s.t$ $\parallel x_0 - 0 \parallel < \delta$ $implies$ $\parallel V(x_0) - V(0) \parallel < \varepsilon$ that is

$$\parallel x_0 \parallel < \delta \ implies \ \parallel V(x_0) \parallel < \varepsilon \qquad (2.3)$$

Let $x(t)$be any solution of equation (2.1) s.t $\parallel x_0 \parallel < \delta$. Since $\dot{V}$ is negative semi-definite i.e. $\dot{V} \leq 0$ then V is non-increasing. This means that if $x \geq x_0$ then

$$V(x) \leq V(x_0) \qquad (2.4)$$

Combining equation (2.2), (2.3) and (2.4) we have that $\parallel x_0 \parallel < \delta$implies $\parallel x \parallel \leq V(x) \leq V(x_0) < \varepsilon \Rightarrow \parallel x \parallel < \delta$. Thus given$\epsilon > 0$ $there\ exists$ $\delta > 0$ $s.t$ $\parallel x_0 \parallel < \delta$ implies $\parallel x(t) \parallel < \varepsilon$. This shows that the trivial solution of $x = 0$ of equation (2.1) is stable.

3.0Results.

1. Krasovskii's Method: This approach assumes the Lyapunov function to be a Hermitian form or quadratic form. We would like to point out here that in [15], such assumption of Hermitian form or quadratic form is unnecessarily restrictive simply because a Hermitian form or a quadratic form may not exist for a given system.We adapt the method used in [16] and extend it to Duffing equation of the form

$$\ddot{x} + c\dot{x} + ax + bx^2 + 2x^3 = p(t) \qquad (3.1)$$

The equivalent systems of equation (3.1) is

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -cx_2 - ax_1 - h(x_1) \qquad (3.2)$$

Where $h(x_1) = bx_1^2 + 2x_1^3$ and $p(t) = 0$.The Lyapunov function associated with this method is given by $V(x) = f(x)^T P f(x)$ where $P = I$ is a symmetric matrix for Krasovskii's theorem and $f(x) = [\dot{x}_1 \quad \dot{x}_2]^T$

The Jacobian matric A $= \left[\frac{\partial f(x)}{\partial x}\right]$ is given by A= $\begin{bmatrix} 0 & 1 \\ -a - h(x_1) & -c \end{bmatrix}$

For asymptotic stability, $A^T P + PA = -Q < 0 \; \forall \; x \in \mathbb{N}$ neighborhood of equilibrium point.

$$A^T P + PA = \begin{bmatrix} 0 & -(a + h(x_1)) \\ 1 & -c \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -(a + h(x_1)) & -c \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -(a + h(x_1)) \\ 1 & -c \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -(a + h(x_1)) & -c \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 - (a + h(x_1)) \\ 1 - (a + h(x_1)) & -2c \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & -1 + (a + h(x_1)) \\ -1 + (a + h(x_1)) & 2c \end{bmatrix} < 0 (3.3)$$

Equation (3.3) shows that the equilibrium point is asymptotically stable.

Moreover, $V(x) = f(x)^T f(x)$

$$= [x_2 \quad -(cx_2 + ax_1 + h(x_1))] \begin{bmatrix} x_2 \\ -(cx_2 + ax_1 + h(x_1)) \end{bmatrix}$$

$$= x_2^2 + (cx_2 + ax_1 + h(x_1))^2 \qquad (3.4)$$

In equation (3.4) $V(x)$ is positive definitesince $V(x) \geq 0$. $x = 0$ is globally asymptotically stable because $V(x) \to \infty$ as$\|x\| \to \infty$

2. Variable Gradient Method: This approach assume that the gradient of the Lyapunov function $V(x)$ is known up to some parameter.To investigate the stability criteria for Lyapunov construction, we must obtain a scalar function V and the time derivative $\dot{V}$ in which state variables are Implicit Functions of time. Using the equivalent systems of equation (3.1)where $a > 0, b > 0$ and $c > 0$

The gradient is in the form$\nabla V(x) = [\nabla V_1 \ . \ . \ . \nabla V_n]^T$

where$\nabla V_i = \sum_{j=1}^{n} h_{ij}x_j$, $i = 1, \ . \ . \ . \ n$ and $\nabla V = \frac{\partial V}{\partial x} = g(x)$

$$\nabla V(x) = \begin{bmatrix} h_{11}x_1 + h_{12}x_2 \\ h_{21}x_1 + h_{22}x_2 \end{bmatrix} \qquad (3.5)$$

Simplifying the coefficient h_{ij} $i,j = 1, \ . \ . \ . \ n$ using the curl condition $\frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial \nabla V_i}{\partial x_j} = \frac{\partial \nabla V_j}{\partial x_i}$

we have$\frac{\partial \nabla V_1}{\partial x_2} = x_1 \frac{\partial h_{11}}{\partial x_2} + x_2 \frac{\partial h_{12}}{\partial x_2} + h_{12}and\frac{\partial \nabla V_2}{\partial x_1} = x_1 \frac{\partial h_{21}}{\partial x_1} + x_2 \frac{\partial h_{22}}{\partial x_1} + h_{12}$

Since $\frac{\partial \nabla V_1}{\partial x_2} = \frac{\partial \nabla V_2}{\partial x_1}$ we have

$$x_1 \frac{\partial h_{11}}{\partial x_2} + x_2 \frac{\partial h_{12}}{\partial x_2} = x_1 \frac{\partial h_{21}}{\partial x_1} + x_2 \frac{\partial h_{22}}{\partial x_1} \qquad (3.6)$$

Obtaining$\dot{V}(x)$ such that $\frac{\partial h_{ik}}{\partial x_j} = 0, i \neq j , k = 1 ,2, \ldots$ and h_{ii} constant, $i = 1, 2, \ . \ . \ .$ we have

$\frac{\partial \nabla V_1}{\partial x_2} = x_1 \frac{\partial h_{11}}{\partial x_2}and\frac{\partial \nabla V_2}{\partial x_1} = x_2 \frac{\partial h_{22}}{\partial x_1}$

$$\dot{V}(x) = \left[\frac{\partial V}{\partial x}\right]^T f(x) = [h_{11}x_1 \quad h_{22}x_2] \begin{bmatrix} x_2 \\ -cx_2 - ax_1 - h(x_1) \end{bmatrix}$$

$$= h_{11}x_1x_2 - h_{22}(cx_2^2 + ax_1x_2 + \ h(x_1)x_2) \qquad (3.7)$$

Integrating$\dot{V}(x)$ and choosing h_{ij} $i , j = 1, \ . \ . \ . \ n$ so that $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite gives

$$V(x) = \int_0^{x_1} \nabla V_1 \, (s_1 , 0) ds_1 + \int_0^{x_2} \nabla V_2 \, (x_1 , s_2) ds_2$$

Since $x_2 = 0$ the first term in equation (3.7) vanishes and we have

$$V(x) = -h_{22}\left[\frac{cx_2^3}{3} + \frac{ax_1x_2^2}{2} + \frac{h(x_1)x_2^2}{2}\right]$$

For $V(x)$ to be positive definite we let $h_{22} = -1$ so that

$$V(x) = \frac{1}{6}[2cx_2^3 + 3ax_1x_2^2 + 3h(x_1)x_2^2] > 0 \qquad (3.8)$$

$$\dot{V}(x) = -[h_{22}(cx_2^2 + ax_1x_2 + h(x_1)x_2) - h_{11}x_1x_2] < 0 \qquad (3.9)$$

Hence (3.8) and (3.9) shows that the equilibrium point is asymptotically stable

3. Cartwright Method: We adapt the method of construction of Lyapunov function used in [17] and extend it to the second order non-linear differential equation of the Duffing type of the form (3.1)**.** In the sequel, Ezeilo and Ogbu [18] asserted that Lyapunov functions are vital in determining stability, instability, boundedness and periodicity of ordinary differential.

Writing the equivalent systems of equation (3.1) in compact form, we have

$$\dot{X} = Ax \qquad (3.10)$$

Where $A = \begin{bmatrix} 0 & 1 \\ -a - h(x) & -c \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}$ (3.11)

The method discussed here is based on the fact that the matrix A defined in equation (3.11) has all its eigenvalues as negative real parts. Then from the general theory which corresponds to any positive quadratic form $U(x)$, there exists another positive definite quadratic form $V(x)$ such that

$$\dot{V} = -U \qquad (3.12)$$

Choosing the most general quadratic form of order two and picking the coefficient in the quadratic form to satisfy equation (3.12) along the solution paths of equation (3.2) we assume

V to be defined by $2V = Ax^2 + By^2 + 2Kxy$ (3.13)

$$\dot{V} = Ax\dot{x} + By\dot{y} + K(\dot{x}y + \dot{y}x)$$

$$= Axy + B\big(-cy - ax - h(x)\big) + Ky^2 + Kx(-cy - ax - h(x))$$

$$= Axy - Bcy^2 - Byax - Byh(x) + Ky^2 - Kcxy - Kax^2 - Kxh(x)$$

Simplifying the coefficients we have

$$\dot{V} = (A - Ba - Kc)xy + (K - Bc)y^2 - (Ka)x^2 - (Kx + By)h(x) \qquad (3.14)$$

Table 1.1: A table showing terms and coefficients of equation (3.14)

Terms	Coefficient
xy	$A - Ba - Kc$
y^2	$K - Bc$
x^2	$-Ka$
$h(x)$	$-(Kx + By)$

To make $\dot{V}$ negative definite, we equate the coefficient of mixed variable to zero and the coefficients of x^2 and y^2 to any positive constant (say δ) i.e.

$$A - Ba - Kc = 0 \qquad \text{(i)}$$

$$K - Bc = \delta \qquad \text{(ii)}$$

$$-Ka = \delta \qquad \text{(iii)}$$

$$-(Kx + By) = 0 \qquad \text{(iv)}$$

From equation (iii) above

$$-Ka = \delta$$

$K = -\frac{\delta}{a}$ (3.15)

Then substituting the value of K into equation (ii) we obtain

$$-\frac{\delta}{a} - Bc = \delta$$

$$-Bc = \delta + \frac{\delta}{a}$$

$B = -\frac{\delta(a+1)}{ca}$ (3.16)

Substituting for K and B in (i) we have

$$A = Ba + Kc$$

$= -\frac{\delta(a+1)}{c} - \frac{\delta c}{a}$ (3.17)

which by further simplification gives that

$A = -\frac{\delta}{ca}[(a+1)a + c^2]$ (3.18)

Substituting for the values of the constant A, B, K in equation (3.13) gives

$$2V = -\frac{\delta}{ca}[(a+1)a + c^2]x^2 - \frac{\delta}{ca}[a+1]y^2 - 2\frac{\delta}{a}xy$$

$V = -\frac{\delta}{2ca}\left[\left((a+1)a + c^2\right)x^2 + (a+1)y^2 + 2cxy\right]$

By choosing $\frac{\delta}{ca} = -1$

$V(x) = \frac{1}{2}\left[\left((a+1)a + c^2\right)x^2 + (a+1)y^2 + 2cxy\right] > 0$ (3.19)

Using equation (3.19) and the fact that $\dot{V} < 0$, the equilibrium point is asymptotically stable

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