Finite Element Technique to the Optimal Control of Two Dimensional Wave Equation with Energy Effect.

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Abstract

This research work considers the optimal state and control of the two dimensional wave equation with energy effect using the Finite Element Technique (FET). The findings in the one dimensional case hold. In addition, the two and three element discretization depict only positive states with negative controls. Other levels of discretization were also considered.

Keywords

Finite Element Technique, Wave Equation, Optimal Control, Optimal State, Element Characteristic Matrix, Differential Equation.

I. Introduction

The research work is an extension of the work of Bawa [1], applicable to the two dimensional optimal control of wave equation with energy effect incorporated with the optimization of a quadratic functional

Min $J(z,u) = \int_0^1 \int_0^1 \int_0^1 [Z^2(x,y,t) + U^2(x,y,t)]\,dxdydt.$

Subject to

$$\frac{\partial^2 z(x,y,t)}{\partial t} + \frac{\partial z(x,y,t)}{\partial t} = \frac{\partial^2 z(x,y,t)}{\partial x^2} + \frac{\partial^2 z(x,y,t)}{\partial y^2} + u(x,y,t)$$

The applications of optimization methods to equations in mathematical physics have been considered [2]. They applied the extended conjugate gradient method to the control problems of diffusion, fluid dynamics and wave propagation. In this work, the finite element technique is used to solve the optimal controlproblem of wave equation with energy effects in two dimensional case.

The finite element technique can be concisely defined as an approximation method of solving complex problems where the solution region is taken as an assemblage of many small – interconnected sub–regions called finite elements.

II.Wave Equation With Energy Effects in Two Dimension

Using the procedural steps in[3] the two – dimensionaloptimal control problem is considered.

The two dimensional wave equations with energy effect is given as

$$\left(\frac{1}{c^2}\right)\frac{\partial^2 z(x,y,t)}{\partial t^2} + \left(\frac{1}{d}\right)\frac{\partial z(x,y,t)}{\partial t} = \frac{\partial^2 z(x,y,t)}{\partial x^2} + \frac{\partial^2 z(x,y,t)}{\partial y^2} \qquad (2.1)$$

The optimization problem under consideration is given by

$$\text{Min J}[Z,U] = \int_0^1\int_0^1\int_0^1 [Z^2(x,y,t) + u^2(x,y,t)]\text{dxdydt} \qquad (2.2)$$

Subject to

$$\frac{\partial^2 z(x,y,t)}{\partial t} + \frac{\partial z(x,y,t)}{\partial t} = \frac{\partial^2 z(x,y,t)}{\partial x^2} + \frac{\partial^2 z(x,y,t)}{\partial y^2} + u(x,y,t)$$

With boundary and initial conditions

$Z(0,y,t) = Z(1,y,t), \quad 0 \le x \le 1$ (2.3)

$Z(x,0,t) = Z(x,1,t), \quad 0 \le y \le 1$

$Z(x,y,0) = Z(x,y,1), \quad 0 \le t \le 1$

Following Singh and Titli[4],the Hamiltonian for (2.2)and(2.3) is given as

$H = Z^2(x,y,t) + U^2(x,y,t) +$
$\lambda^T\left[Z_{XX}(x,y,t) + Z_{yy}(x,y,t) + U(x,y,t)\right]$ (2.4)

Where $\lambda^T = \lambda^T(t)$

Setting

$f(z,u) = Z_{XX}(x,y,t) + Z_{yy}(x,y,t) + U(x,y,t)$

$g(z,u) = Z^2(x,y,t) + U^2(x,y,t)$

The first order necessary conditions for optimality is

$Z_t(x,y,t) = H_\lambda(x,y,t)$

$= Z_{XX}(x,y,t) + Z_{yy}(x,y,t) + U(x,y,t)$

$= f\big(Z(x,y,t), U(x,y,t)\big)$ (2.5)

$\lambda_t = -H_z = [f_z]^T\lambda - g_z = -2z(x,y,t)$ (2.6)

$$H_u = 0$$

$\text{or}[f_u] \quad \lambda^T + g_u = 0$ (2.7)

Where $H = g(z,u) + \lambda^T(t)f(z,u)$

Equation (2.7) gives $\lambda + 2u = 0$ or $\lambda = -2u$

Equations (2.6) and (2.7) give

$$\lambda_t = 2u_t(x,y,t) = -2z(x,y,t)$$

Hence,

$$Z(x,y,t) = -U_t(x,y,t) \qquad (2.8)$$

Assuming(2.8)as a Fourier solution proposed by Ibiejugba [5] and Duchateau and Zachmann[6]

$$Z(x,y,t) = \sum_{i=1}^{\infty} \alpha_i\,(t) \sin \pi i\, x \sin \pi i y \ (2.9)$$

$U(x,y,t)$

$$= \sum_{i=1}^{\infty} u_i\,(t) \sin \pi i\, x \sin \pi i y \tag{2.10}$$

Thisgives the new solution as:

$$Z(x,y,t) = \sum_{i=1}^{\infty} U_{it}\ \sin \pi i\, x \sin \pi i y \tag{2.11}$$

It then follows that

$$\alpha_i(t) = U_{it}(t)$$

and

$$Z_t(x,y,t) = \sum_{i=1}^{\infty} U_{itt}\,(t) \sin \pi i\, x \sin \pi i y$$

$$Z_{tt}(x,y,t) = \sum_{i=1}^{\infty} U_{itt}\,(t) \sin \pi i\, x \sin \pi i y$$

$$Z_{xx}(x,y,t) = \sum_{i=1}^{\infty} i^2(\pi^2)U_{itt}\,(t) \sin \pi i\, x \sin \pi i y$$

$$Z_{yy}(x,y,t) = \sum_{i=1}^{\infty} i^2(-\pi^2)U_i\,(t) \sin \pi i\, x \sin \pi i y$$

$$Z(x,y,0) = \sum_{i=1}^{\infty} U_{it}\,(0) \sin \pi i\, x \sin \pi i y$$

The constrained equation gives

$$Z_{tt}(x,y,t) + Z_t(x,y,t) = Z_{xx}(x,y,t) + Z_{yy}(x,y,t) + U(x,y,t)$$

Hence

$$\sum_{i=1}^{\infty} U_{ittt}\,(t) \sin \pi i\, x \sin \pi i y + \sum_{i=1}^{\infty} U_{itt}\,(t) \sin \pi i\, x \sin \pi i y =$$

$$\sum_{i=1}^{\infty} -i^2\pi^2 U_{it}\,(t) \text{ sin πi x sin πiy} + \sum_{i=1}^{\infty} -i^2\pi^2 U_{it}\,(t) \text{ sin πi x sin πiy}$$

This implies that

$$U_{ittt}(t) + U_{itt}(t) = -i^2\pi^2 u_{it}(t) - i^2\pi^2 u_{it}(t) + u_i(t)$$

$$U_{ittt}(t) + U_{itt}(t) = -2i^2\pi^2 u_{it}(t) + u_i(t)$$

$$U_{ittt}(t) - U_{itt}(t) = -2i^2\pi^2 u_{it}(t) + u_i(t)$$

and the problem can be written in the form

$$\text{Min}\int_0^1 [u_1^2 + u_2^2 + \cdots \dots \dots + u_n{}^2]dt + \int_0^1 [u_{it}^2 + u_{2t}^2 + \cdots \dots \dots + u_n{}^2]dt \qquad (2.12)$$

The corresponding unconstrained problem is given by

$$U_{ittt}(t) - u_{1tt} - 2\pi^2 1^2 u_{it} + u_1$$

$$U_{2ttt}(t) = -u_{2tt} - 2\pi^2 2^2 u_{2t} + u_2$$

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$$U_{nttt}(t) = -U_{ntt} - 2\pi^2 n^2 u_{nt} + u_n \qquad (2.13)$$

III.Idealisation

System (2.13)is now solved using the finite element technique by discretizing the domain $[t = 0 \text{ to } 1]$with elements of equal length.Let the nodes be denoted by i andj and the nodal values of the field variable$u$ by $u_i$and $u_j$and nodal conditions

$$U(t) = U_i \text{at } t = t_i$$

$$U(t) = U_j \text{at } t = t_j \qquad (3.1)$$

After further steps and re-arrangement of terms gives

$$U(t) = [\text{N}(t)]\vec{U}^{(e)} \qquad (3.2)$$

Where $\text{N}_i(t) = t_j - {}^{t}/_{\text{L}}$,

$$N_j(t) = t - {t_i}/L$$

and $\vec{U}^{(e)} = \begin{pmatrix} U_i \\ U_j \end{pmatrix}$, that is vector of nodal unknowns of element e.

IV. Element Characteristic Matrices and Vectors

Since the sequence of equation (2.13)only differ by constant multiplicants,it suffices to solve just one of the n-third order equations.Choosing

$U\mathrm{n}_{ttt}(t) = -U\mathrm{n}_{tt} - 2\pi^2 n^2 U\mathrm{n}_1 + U\mathrm{n}$

an equivalent variational problem is given as an optimization problem

$$\text{Minimize } I = \frac{1}{2}\int_0^1 \left[-\left[\frac{\partial^2 u_n}{\partial t^2}\right]^2 + \left[\frac{\partial U_n}{\partial t}\right]^2 + 4\pi^2 n^2 U_n - Un^2\right] dt \qquad (4.1)$$

The element characteristic matrices and vectors can be identified by expressing the functional I in matrix form. Evaluating the integral in I over the length of element e, gives

$$I^{(e)} = \frac{1}{2}\int_0^1 \left[-\left[\frac{\partial^2 u_n}{\partial t^2}\right]^2 + \left[\frac{\partial U_n}{\partial t}\right]^2 + 4\pi^2 n^2 U_n - Un^2\right] dt \qquad (4.2)$$

Expressing the functional I as a sum of E elemental quantities $I^{(e)}$ gives:

$$I = \sum_{e=I}^{E} I^{(e)}$$

Where

$$I^{(e)} = \frac{1}{2}\int_{t_i}^{t_j} \left[-\left[\frac{\partial^2 u_n}{\partial t^2}\right]^2 + \left[\frac{\partial U_n}{\partial t}\right]^2 + 4\pi^2 n^2 U_n - Un^2\right] dt \qquad (4.3)$$

By substituting (3.2) into (4.3) gives

$$\mathrm{I}^{(\mathrm{e})} = \frac{1}{2}\int_{t_i}^{t_j}\Bigg[-U_n^{\rightarrow(e)T}\left[\frac{\partial^2 N}{\partial t^2}\right]^T\left[\frac{\partial^2 N}{\partial t^2}\right]U_n^{\rightarrow(e)} + U_n^{\rightarrow(e)T}\left[\frac{\partial N}{\partial t}\right]^T\left[\frac{\partial N}{\partial t}\right]U_n^{\rightarrow(e)} +$$

$$4\pi^2 n^2[N]U_n^{\rightarrow(e)} - U_n^{(e)^T}[N]^T[N]U_n^{\rightarrow(e)}\Bigg]\,dt \qquad (4.4)$$

For the stationeries of I, we use the necessary conditions

$$\frac{\partial I}{\partial U_i} = \sum_{e=\mathrm{I}}^{E}\frac{\partial I^{(\mathrm{e})}}{\partial U_i} = 0, \qquad (4.5)$$

i=1, 2…………….M

Where E is the number of elements and M is the number of nodal degrees of freedom. Equation (4.5) can also be expressed as

$$\sum_{e=1}^{E}\frac{\partial I^{(e)}}{\partial U^{\rightarrow(e)}} = 0$$

i.e

i.e

$$\sum_{e=\mathrm{I}}^{E}\left[k^{(e)}\right]U_n^{\rightarrow(e)} = \sum_{e=\mathrm{I}}^{E}\vec{\mathrm{P}}^{(e)} \qquad (4.6)$$

Where $\mathrm{K}^{(\mathrm{e})}$ = element characteristic matrix

(4.7)

$\vec{\mathrm{P}}^{(\mathrm{e})}$ = element characteristic vector

$$= \int_{t_i}^{t_j} 2\pi^2\, n^2 [N]^T dt \qquad (4.8)$$

V. Assemblage of Element Characteristic Matrices and Vectors.

The element characteristic matrices and vectors are now assembled to obtain the overall equations as:

$$[K]U_n{}^{\rightarrow} = P^{\rightarrow} \qquad (5.1)$$

Case E = 2

$$\frac{1}{12}\begin{bmatrix} -22 & 25 & 0 \\ 25 & -44 & 25 \\ 0 & 25 & -22 \end{bmatrix}\begin{bmatrix} U_{n_1} \\ U_{n_2} \\ U_{n_3} \end{bmatrix} = \frac{\pi^2 n^2}{2}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Case E=3

$$\frac{1}{18}\begin{bmatrix} -52 & 55 & 0 & 0 \\ 55 & -104 & 55 & 0 \\ 0 & 55 & -104 & 55 \\ 0 & 0 & 55 & -52 \end{bmatrix}\begin{bmatrix} U_{n_1} \\ U_{n_2} \\ U_{n_3} \\ U_{n_4} \end{bmatrix} = \frac{\pi^2 n^2}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

Case E=4

$$\frac{1}{24}\begin{bmatrix} -94 & 97 & 0 & 0 & 0 \\ 97 & -188 & 97 & 0 & 0 \\ 0 & 97 & -188 & 97 & 0 \\ 0 & 0 & 97 & -188 & 97 \\ 0 & 0 & 0 & 97 & -94 \end{bmatrix}\begin{bmatrix} U_{n_1} \\ U_{n_2} \\ U_{n_3} \\ U_{n_4} \\ U_{n_5} \end{bmatrix} = \frac{\pi^2 n^2}{4}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

Case E =5

$$\frac{1}{30}\begin{bmatrix} -148 & 151 & 0 & 0 & 0 & 0 \\ 151 & -296 & 151 & 0 & 0 & 0 \\ 0 & 151 & -296 & 151 & 0 & 0 \\ 0 & 0 & 151 & -296 & 151 & 0 \\ 0 & 0 & 0 & 151 & -296 & 151 \\ 0 & 0 & 0 & 0 & 151 & -148 \end{bmatrix}\begin{bmatrix} U_{n_1} \\ U_{n_2} \\ U_{n_3} \\ U_{n_4} \\ U_{n_5} \\ U_{n_6} \end{bmatrix} = \frac{\pi^2 n^2}{5}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

`Case E =6

$$\frac{1}{36}\begin{bmatrix} -214 & 217 & 0 & 0 & 0 & 0 & 0 \\ 217 & -428 & 217 & 0 & 0 & 0 & 0 \\ 0 & 217 & -428 & 217 & 0 & 0 & 0 \\ 0 & 0 & 217 & -428 & 0 & 0 & 0 \\ 0 & 0 & 0 & 217 & 217 & 217 & 0 \\ 0 & 0 & 0 & 0 & -428 & -428 & 217 \\ 0 & 0 & 0 & 0 & 217 & 217 & -214 \end{bmatrix}\begin{bmatrix} U_{n_1} \\ U_{n_2} \\ U_{n_3} \\ U_{n_4} \\ U_{n_5} \\ U_{n_6} \\ U_{n_7} \end{bmatrix} = \frac{\pi^2 n^2}{6}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

Case E =7

$$\frac{1}{42}\begin{bmatrix} -292 & 295 & 0 & 0 & 0 & 0 & 0 & 0 \\ 295 & -584 & 295 & 0 & 0 & 0 & 0 & 0 \\ 0 & 295 & -584 & 295 & 0 & 0 & 0 & 0 \\ 0 & 0 & 295 & -584 & 295 & 0 & 0 & 0 \\ 0 & 0 & 0 & 295 & -584 & 295 & 0 & 0 \\ 0 & 0 & 0 & 0 & 295 & -584 & 295 & 0 \\ 0 & 0 & 0 & 0 & 0 & 295 & -584 & 295 \\ 0 & 0 & 0 & 0 & 0 & 0 & 295 & -292 \end{bmatrix}\begin{bmatrix} U_{n_1} \\ U_{n_2} \\ U_{n_3} \\ U_{n_4} \\ U_{n_5} \\ U_{n_6} \\ U_{n_7} \\ U_{n_8} \end{bmatrix} = \frac{\pi^2 n^2}{7}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

Case E =8

$$\frac{1}{48}\begin{bmatrix} -382 & 385 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 385 & -764 & 385 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 385 & -764 & 385 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 385 & -764 & 385 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 385 & -764 & 385 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 385 & -764 & 385 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 385 & -764 & 385 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 385 & -764 & 385 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 385 & -382 \end{bmatrix}\begin{bmatrix} U_{n_1} \\ U_{n_2} \\ U_{n_3} \\ U_{n_4} \\ U_{n_5} \\ U_{n_6} \\ U_{n_7} \\ U_{n_8} \\ Un_9 \end{bmatrix} = \frac{\pi^2 n^2}{8}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

Case E =9

$$\frac{1}{54}\begin{bmatrix} -484 & 487 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 487 & -968 & 487 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 487 & -968 & 487 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 487 & -968 & 487 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 487 & -968 & 487 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 487 & -968 & 487 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 487 & -968 & 487 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 487 & -968 & 487 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 487 & -968 & 487 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 487 & -484 \end{bmatrix}\begin{bmatrix} U_{n_1} \\ U_{n_2} \\ U_{n_3} \\ U_{n_4} \\ U_{n_5} \\ U_{n_6} \\ U_{n_7} \\ U_{n_8} \\ U_{n_9} \\ U_{n_{10}} \end{bmatrix} = \frac{\pi^2 n^2}{9}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

Case E =10

$$\frac{1}{60}\begin{bmatrix} -598 & 601 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 601 & -1196 & 601 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 601 & -1196 & 601 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 601 & -1196 & 601 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 601 & -1196 & 601 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 601 & -1196 & 601 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 601 & -1196 & 601 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 601 & -1196 & 601 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 601 & -1196 & 601 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 601 & -1196 & 601 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 601 & -598 \end{bmatrix}\begin{bmatrix} U_{n_1} \\ U_{n_2} \\ U_{n_3} \\ U_{n_4} \\ U_{n_5} \\ U_{n_6} \\ U_{n_7} \\ U_{n_8} \\ U_{n_9} \\ U_{n_{10}} \\ U_{n_{11}} \end{bmatrix} = \frac{\pi^2 n^2}{10}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

VI. Computational Results.

The computational results for the two dimensional wave equation with energy effects after incorporating the boundary conditions is given as

Case E = 2

$$\begin{bmatrix} U_{n_1} \\ U_{n_2} \\ U_{n_3} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.2727273\pi^2 n^2 \\ 0 \end{bmatrix} \begin{bmatrix} Z_{n_1} \\ Z_{n_2} \\ Z_{n_3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.54\,5\,4546\pi^2 n^2 \\ 0 \end{bmatrix}$$

Case E = 3

$$\begin{bmatrix} U_{n_1} \\ U_{n_2} \\ U_{n_3} \\ U_{n_4} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.244898\pi^2 n^2 \\ -0.244898\pi^2 n^2 \\ 0 \end{bmatrix} \begin{bmatrix} Z_{n_1} \\ Z_{n_2} \\ Z_{n_3} \\ Z_{n_4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.734694\,n^2 \\ 0 \\ 0 \end{bmatrix}$$

Case E = 4

$$\begin{bmatrix} U_{n_1} \\ U_{n_2} \\ U_{n_3} \\ U_{n_4} \\ U_{n_5} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.2069466\pi^2 n^2 \\ -0.27738\pi^2 n^2 \\ -0.2069466\pi^2 n^2 \\ 0 \end{bmatrix} \begin{bmatrix} Z_{n_1} \\ Z_{n_2} \\ Z_{n_3} \\ Z_{n_4} \\ Z_{n_5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.8277864\pi^2 n^2 \\ 0.2817376\pi^2 n^2 \\ -0.2817376\pi^2 n^2 \\ 0 \end{bmatrix}$$

Case E = 5

$$\begin{bmatrix} U_{n_1} \\ U_{n_2} \\ U_{n_3} \\ U_{n_4} \\ U_{n_5} \\ U_{n_6} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.1765495\pi^2 n^2 \\ -0.2666137\pi^2 n^2 \\ -0.2666137\pi^2 n^2 \\ -0.1765495\pi^2 n^2 \\ 0 \end{bmatrix} \begin{bmatrix} Z_{n_1} \\ Z_{n_2} \\ Z_{n_3} \\ Z_{n_4} \\ Z_{n_5} \\ Z_{n_6} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.8827475\pi^2 n^2 \\ 0.450321\pi^2 n^2 \\ 0 \\ -0.450321\pi^2 n^2 \\ 0 \end{bmatrix}$$

Case E = 6

$$\begin{bmatrix} U_{n_1} \\ U_{n_2} \\ U_{n_3} \\ U_{n_4} \\ U_{n_5} \\ U_{n_6} \\ U_{n_7} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.1531537\pi^2n^2 \\ -0.2467732\pi^2n^2 \\ -0.27827\pi^2n^2 \\ -0.2467732\pi^2n^2 \\ -0.1531537\pi^2n^2 \\ 0 \end{bmatrix} \begin{bmatrix} Z_{n_1} \\ Z_{n_2} \\ Z_{n_3} \\ Z_{n_4} \\ Z_{n_5} \\ Z_{n_6} \\ Z_{n_7} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.9189222\pi^2n^2 \\ 0.561717\pi^2n^2 \\ 0.1889808\pi^2n^2 \\ -0.1889808\pi^2n^2 \\ -0.561717\pi^2n^2 \\ 0 \end{bmatrix}$$

Case E = 7

$$\begin{bmatrix} U_{n_1} \\ U_{n_2} \\ U_{n_3} \\ U_{n_4} \\ U_{n_5} \\ U_{n_6} \\ U_{n_7} \\ U_{n_8} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.1349293\pi^2n^2 \\ -0.2264363\pi^2n^2 \\ -0.2726599\pi^2n^2 \\ -0.2726599\pi^2n^2 \\ -0.2264363\pi^2n^2 \\ -0.1349293\pi^2n^2 \\ 0 \end{bmatrix} \begin{bmatrix} Z_{n_1} \\ Z_{n_2} \\ Z_{n_3} \\ Z_{n_4} \\ Z_{n_5} \\ Z_{n_6} \\ Z_{n_7} \\ Z_{n_8} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.944505\pi^2n^2 \\ 0.640549\pi^2n^2 \\ 0.3235652\pi^2n^2 \\ 0 \\ -0.3235652\pi^2n^2 \\ -0.640549\pi^2n^2 \\ 0 \end{bmatrix}$$

Case E = 8

$$\begin{bmatrix} U_{n_1} \\ U_{n_2} \\ U_{n_3} \\ U_{n_4} \\ U_{n_5} \\ U_{n_6} \\ U_{n_7} \\ U_{n_8} \\ U_{n_9} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.1204426\pi^2n^2 \\ -0.2078394\pi^2n^2 \\ -0.2608282\pi^2n^2 \\ -0.2785834\pi^2n^2 \\ -0.2608282\pi^2n^2 \\ -0.20778394\pi^2n^2 \\ -0.1204426\pi^2n^2 \\ 0 \end{bmatrix} \begin{bmatrix} Z_{n_1} \\ Z_{n_2} \\ Z_{n_3} \\ Z_{n_4} \\ Z_{n_5} \\ Z_{n_6} \\ Z_{n_7} \\ Z_{n_8} \\ Z_{n_9} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.965408\pi^2n^2 \\ 0.6991744\pi^2n^2 \\ 0.4239104\pi^2n^2 \\ 0.1420416\pi^2n^2 \\ -0.1420416\pi^2n^2 \\ -0.4239104\pi^2n^2 \\ -0.6991744\pi^2n^2 \\ 0 \end{bmatrix}$$

Case E = 9

$$\begin{bmatrix} U_{n_1} \\ U_{n_2} \\ U_{n_3} \\ U_{n_4} \\ U_{n_5} \\ U_{n_6} \\ U_{n_7} \\ U_{n_8} \\ U_{n_9} \\ U_{n_{10}} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.1086946\pi^2 n^2 \\ -0.1914094\pi^2 n^2 \\ -0.2471253\pi^2 n^2 \\ -0.2751559\pi^2 n^2 \\ -0.2751559\pi^2 n^2 \\ -0.2471253\pi^2 n^2 \\ -0.1914094\pi^2 n^2 \\ -0.1086946\pi^2 n^2 \\ 0 \end{bmatrix} \begin{bmatrix} Z_{n_1} \\ Z_{n_2} \\ Z_{n_3} \\ Z_{n_4} \\ Z_{n_5} \\ Z_{n_6} \\ Z_{n_7} \\ Z_{n_8} \\ Z_{n_9} \\ Z_{n_{10}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.9782514\pi^2 n^2 \\ 0.7444332\pi^2 n^2 \\ 0.5014431\pi^2 n^2 \\ 0.2522754\pi^2 n^2 \\ 0 \\ -0.2522754\pi^2 n^2 \\ -0.5014431\pi^2 n^2 \\ -0.7444332\pi^2 n^2 \\ 0 \end{bmatrix}$$

Case E = 10

$$\begin{bmatrix} U_{n_1} \\ U_{n_2} \\ U_{n_3} \\ U_{n_4} \\ U_{n_5} \\ U_{n_6} \\ U_{n_7} \\ U_{n_8} \\ U_{n_9} \\ U_{n_{10}} \\ U_{n_{11}} \end{bmatrix} = \begin{bmatrix} 0 \\ -9.899617E-02\pi^2 n^2 \\ -0.1770373\pi^2 n^2 \\ -0.2333442\pi^2 n^2 \\ -0.2673548\pi^2 n^2 \\ -0.2787295\pi^2 n^2 \\ -0.2673548\pi^2 n^2 \\ -0.2333442\pi^2 n^2 \\ -0.1770373\pi^2 n^2 \\ -9.899617E-02\pi^2 n^2 \\ 0 \end{bmatrix} \begin{bmatrix} Z_{n_1} \\ Z_{n_2} \\ Z_{n_3} \\ Z_{n_4} \\ Z_{n_5} \\ Z_{n_6} \\ Z_{n_7} \\ Z_{n_8} \\ Z_{n_9} \\ Z_{n_{10}} \\ Z_{n_{11}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.9899617\pi^2 n^2 \\ 0.7804113\pi^2 n^2 \\ 0.563069\pi^2 n^2 \\ 0.340106\pi^2 n^2 \\ 0.113747\pi^2 n^2 \\ -0.113747\pi^2 n^2 \\ -0.340106\pi^2 n^2 \\ -0.563069\pi^2 n^2 \\ -0.7804113\pi^2 n^2 \\ 0 \end{bmatrix}$$

VII. Conclusion

Expressions for the optimal state $Z(x,y)$and the optimal control $U(x,y)$for the two dimensional wave equation with energy effect was derived.Two and three elements discretization depict only positive state with negative control.As the number of elements increases,the controls get smaller and the states shrink at the second node.This research will enhance further computational processes using the FET towards the derivation of the optimal controls for states at various spatial planes by researchers.

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