A Method for Solving Interval Systems of Linear Equations

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ABSTRACT

The system of linear equation has a great importance in many real life problems such as economics, optimization and in various engineering field. We know that system of linear equations, in general is solved for crisp unknowns. In actual case the parameters of the system of linear equations are modeled by taking the experimental or observational data. So the parameters of the system actually contain uncertainty rather than the crisp one. The uncertainties may be considered in term of interval number.

Recently different authors have investigated these problems by various methods. Although solutions obtained by these methods are good but sometimes the method requires lengthy procedure and computationally not efficient.

In this paper we propose an exact method for solving interval system of linear equation. We have tested the method and it is producing a better result in comparison with the existing ones.

1.0 INTRODUCTION

System of linear equation can be solved by different methods. If the coefficient matrix and right vector are in crisp then this system can be solved by the Cramer's rule, Gauss- Elimination method, Gauss- Jacobi iteration method, Gauss- seidel iteration method, matrix inversion method, matrix factorization method etc. When the coefficient matrix and the right vector are in the interval form then also we may find the solution by Gauss-Elimination type method (interval form).In this context different authors develop different methods. Kolev [1] developed a method for outer interval solution of linear parametric systems. Skalna [2] introduced methods for solving systems of linear equations of structure mechanics with interval parameters. Popova [3] also introduced a method on the solution of parameterized linear systems. Walter Kraemer [4] investigated computing and visualizing solution sets of interval linear systems. Diptiranjan and Chakraverty recently found a numerical solution of interval and fuzzy system of linear equation[5]. However, in this work, we reviewe the methods presented bySuparna Dasand S. Chakraverty[6], Shohreh Abolmasaumi and Majid Alavi [7], Abolmasaumi [8] and the Gaussian elimination methods using modified interval arithmetic proposed by Mehdi Allahdadi and Zohreh Khorran [9]. Finally we compare our results with those obtained in the reviewed papers. All the methods are aimed at finding the narrowest possible values of the lower and upper bounds of the interval solutions x in the given systems of interval linear equations. In this paper we have proposed a new method for solving interval system of linear equations which is undoubtedly a good method for solving this type of system of equations and always gives strong solution.On the other hand, applying the proposedmethod the interval width obtained is less than the interval width obtained bythe existing methods. Moreover, the closeness of interval width in system of equations tothe right hand side is important so the proposed method is more effective and efficient forsolving interval linear system of equations.

2.0 BASICS CONCEPTS AND NOTATIONS OF INTERVAL ARITHMETIC.

Interval will be represented by boldface brackets "**[ ]**", used for intervals defined by an upper bound and a lower bound. Underscore will be used to denote lower bounds of intervals and over score will denote the upper bounds

Definition 2.1

An interval number X ∈ R*is* generally represented as $[\underline{X}, \overline{X}]$ where $\underline{X} \leq \overline{X}$.

If $\underline{A}$and $\overline{A}$are two matrices in $R^{n \times n}$and $\underline{A} \leq \overline{A}$, then the set of matrices

$$A^I = [\underline{A}; \overline{A}] = \{A \in R : \underline{A} \leq A \leq \overline{A}\};$$

is called an interval matrix, and the matrices $\underline{A}$and $\overline{A}$are called lower and upper bounds of A^I.

Definition 2.2

A square interval matrix $A^I$is called regular if each $A \in A^I$is nonsingular.

Definition 2.3

Interval arithmetic operations are defined on R such that the interval result encloses all possible results. The four standard operations (+,-, ·, ÷) are defined over interval numbers as follows. If $X = [\underline{x}, \overline{x}]$ and $Y = [\underline{y}, \overline{y}]$ then,

$$X + Y = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$$

$$X - Y = [\underline{x} - \overline{y}, \overline{x} - \underline{y}]$$

$$X \cdot Y = [\min\{\underline{xy}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{xy}\} \max\{\underline{xy}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{xy}\}]$$

$$X \div Y = [\underline{x}, \overline{y}] \cdot \left[\frac{1}{\overline{y}}, \frac{1}{\underline{y}}\right] \quad \text{such that } 0 \notin [\underline{y}, \overline{y}].$$

3.0 RESEARCH METHOD

Let us consider the system of linear equation.

$$A^I x^I = b^I \qquad (1)$$

Where $A^I = [\underline{A}, \overline{A}]$ is an interval matrix of order *n* and $b^I = [\underline{b}, \overline{b}]$is an interval vector of *n* components. This linear system is called interval linear system.We can also write a linear system of interval equations (1) explicitly as given below.

$$A^I = \begin{bmatrix} (\underline{a}_{11}, \overline{a}_{11}) & (\underline{a}_{12}, \overline{a}_{12}) \\ (\underline{a}_{21}, \overline{a}_{21}) & (\underline{a}_{22}, \overline{a}_{22}) \end{bmatrix}, X^I = \begin{bmatrix} (\underline{x}_1, \overline{x}_1) \\ (\underline{x}_2, \overline{x}_2) \end{bmatrix} and b^I = \begin{bmatrix} (\underline{b}_1, \overline{b}_1) \\ (\underline{b}_2, \overline{b}_2) \end{bmatrix} \qquad (2)$$

Where

$$\underline{A} = \begin{bmatrix} \underline{a}_{11} & \underline{a}_{12} \\ \underline{a}_{21} & \underline{a}_{22} \end{bmatrix}, \underline{X} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} \text{ And } \underline{b} = \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \end{bmatrix} \qquad (3)$$

and

$$\overline{A} = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{12} \\ \overline{a}_{21} & \overline{a}_{22} \end{bmatrix}, \overline{X} = \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix} and \overline{b} = \begin{bmatrix} \overline{b}_1 \\ \overline{b}_2 \end{bmatrix} \qquad (4)$$

Where by the members in an interval are both either positive or negative.Now interval values are operated through the interval arithmetic rules. Because of the more computation involved in this procedure it is a difficult task to perform and the margin of uncertainty increases drastically. To overcome the above difficulties (to a certain extent) we now propose a new methodby solving the lower and upper bounds of the system separately.

Definition 3.1:

If A^I is regular (i.e $0 \notin |A|$), then A^I is invertible.

If the inverse of A^I exists then the systems (1) will be written as

$$A^{I-1}A^I X = A^{-1}b^I$$

$$X^I = A^{I-1}b^I$$

But from the elementary matrix we know that

$$A^{I-1} = \frac{1}{|A^I|} adjA^I$$

Proposition 3.2: [10]

If $A^I = [\, \underline{A}\,, \overline{A}\,] \in \mathcal{R}^{nxn}$ where $\underline{A} and \overline{A}$ are the lower and upper bounds of the interval A. If $\underline{A} and \overline{A}$ are regular and $\underline{A}^{-1} \geq 0$, $\overline{A}^{-1} \geq 0$, then A^I is regular and

$$A^{I-1} = [\overline{A}^{-1}, \underline{A}^{-1}] \geq 0$$

Theorem 3.3:

If $\underline{A} and \overline{A}$ are regular then their inverses exists and the solution to $A^I X^I = b^I$ will be

$$\frac{\underline{x}_1}{\underline{x}_2} = \frac{1}{|\underline{A}|}\begin{bmatrix} \underline{a}_{22}\underline{b}_1 & -\underline{a}_{12}\underline{b}_2 \\ \underline{a}_{21}\underline{b}_1 & \underline{a}_{11}\underline{b}_2 \end{bmatrix}$$

and

$$\frac{\overline{x}_1}{\overline{x}_2} = \frac{1}{|\overline{A}|}\begin{bmatrix} \overline{a}_{22}\overline{b}_1 & -\overline{a}_{12}\overline{b}_2 \\ -\overline{a}_{21}1 & \overline{a}_{11}\overline{b}_2 \end{bmatrix}$$

Proof:

$$\text{Let}\underline{A} = \begin{bmatrix} \underline{a}_{11} & \underline{a}_{12} \\ \underline{a}_{21} & \underline{a}_{22} \end{bmatrix}, \underline{X} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} \text{ And } \underline{b} = \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \end{bmatrix}$$

If $\underline{A}$is regular (i.e $|\underline{A}| \geq 0$) then $\underline{A}$ *is invertible* that is inverse of $\underline{A}$exists.Then, we have,

$$\underline{A}^{-1}\underline{AX} = \underline{A}^{-1}\underline{b}$$

$\underline{X} = \underline{A}^{-1}\underline{b}$

From the concepts of matrix we know that $\underline{A}^{-1} = \frac{1}{|\underline{A}|} adj\underline{A}$ where $adj\underline{A} = \begin{bmatrix} \underline{a}_{22} & -\underline{a}_{12} \\ \underline{a}_{21} & \underline{a}_{11} \end{bmatrix}$

$$\underline{X} = \frac{1}{|\underline{A}|}\begin{bmatrix} \underline{a}_{22} & -\underline{a}_{12} \\ \underline{a}_{21} & \underline{a}_{11} \end{bmatrix}\begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \end{bmatrix} \text{which implies that}$$

$$\frac{\underline{x}_1}{\underline{x}_2} = \frac{1}{|\underline{A}|}\begin{bmatrix} \underline{a}_{22}\underline{b}_1 & -\underline{a}_{12}\underline{b}_2 \\ \underline{a}_{21}\underline{b}_1 & \underline{a}_{11}\underline{b}_2 \end{bmatrix}$$

Hence $\underline{x}_1 = \frac{1}{|\underline{A}|}(\underline{a}_{22}\underline{b}_1 + (-\underline{a}_{12}\underline{b}_2)$and $\underline{x}_2 = \frac{1}{|\underline{A}|}(\underline{a}_{21}\underline{b}_1 + (-\underline{a}_{11}\underline{b}_2)$

The proof of $\overline{X}$follows similalry as$\underline{X}$.

4.0 RESULTS AND ANALYSIS

Example 4.1: Let us consider an example given in [7].

$$\begin{bmatrix} [3,4] & [1,2] \\ [0,1] & [7,8] \end{bmatrix} \begin{bmatrix} [\underline{x}_1, \overline{x}_1] \\ [\underline{x}_2, \overline{x}_2] \end{bmatrix} = \begin{bmatrix} [2,4] \\ -1,1] \end{bmatrix}$$

Using theirmethods and techniques, Abolmasoumiand Alavi . [7]obtained the solution of the above system as$\underline{x}_1, \underline{x}_2 = [0.6231, -0.2075]$and $\overline{x}_1, \overline{x}_2 = [1.2083\ , -0.0520]$.

When we apply the proposed method we have,

$$\begin{bmatrix} [3,4] & [1,2] \\ [0,1] & [7,8] \end{bmatrix} \begin{bmatrix} [\underline{x}_1, \overline{x}_1] \\ [\underline{x}_2, \overline{x}_2] \end{bmatrix} = \begin{bmatrix} [2,4] \\ -1,1] \end{bmatrix}$$

$$\underline{A}^I = \begin{bmatrix} 3 & 1 \\ 0 & 7 \end{bmatrix},\ \underline{X}^I = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix} \quad \text{and} \quad \underline{b}^I = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\underline{A}^* = \begin{bmatrix} 7 & -1 \\ -0 & 3 \end{bmatrix} \ and \ |\underline{A}| = 21 + 0 = 21$$

$$\begin{bmatrix} \underline{X}_1 \\ \underline{X}_2 \end{bmatrix} = \frac{1}{|\underline{A}|}[\underline{A}^*][\underline{b}^I]$$

$$\begin{bmatrix} \underline{X}_1 \\ \underline{X}_2 \end{bmatrix} = \frac{1}{21}\begin{bmatrix} 7 & -1 \\ -0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} \underline{X}_1 \\ \underline{X}_2 \end{bmatrix} = \frac{1}{21}\begin{bmatrix} 14 + 1 \\ -0 - 3 \end{bmatrix}$$

$$\begin{bmatrix} \underline{X}_1 \\ \underline{X}_2 \end{bmatrix} = \frac{1}{21}\begin{bmatrix} 15 \\ -3 \end{bmatrix}$$

$\underline{X}_1 = 0.7142$ And $\underline{X}_2 = -0.14285$

Similarly for the maximum values we have

$$\overline{A}^I = \begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix}, \overline{X}^I = \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix} \ and \ \overline{b}^I = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\overline{A}^* = \begin{bmatrix} 8 & -2 \\ -1 & 4 \end{bmatrix} \ and \ |\overline{A}| = 32 - 2 = 30$$

$$\begin{bmatrix} \overline{X}_1 \\ \overline{X}_2 \end{bmatrix} = \frac{1}{|\overline{A}|}[\overline{A}^*][\overline{b}^I]$$

$$\begin{bmatrix} \overline{X}_1 \\ \overline{X}_2 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} 8 & -2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix}\overline{X}_1\\ \overline{X}_2\end{bmatrix} = \frac{1}{30}\begin{bmatrix}32-2\\ -4+4\end{bmatrix}$$

$$\begin{bmatrix}\overline{X}_1\\ \overline{X}_2\end{bmatrix} = \frac{1}{30}\begin{bmatrix}30\\ 0\end{bmatrix}$$

$$\overline{X}_1 = 1 \quad \text{And } \overline{X}_2 = 0$$

The solution of the system is then $\underline{x}_1, \underline{x}_2 = [0.7142, -0.1429]$ and $\overline{x}_1, \overline{x}_2 = [1, 0]$

Example 4.2: Let us consider another example given in[6].

$$\begin{bmatrix}[2,3] & [4,5]\\ [3,4] & [2,3]\end{bmatrix}\begin{bmatrix}[\underline{x}_1, & \overline{x}_1]\\ [\underline{x}_2, & \overline{x}_2]\end{bmatrix} = \begin{bmatrix}[1, & 2]\\ [18, & 19]\end{bmatrix}$$

Applying the proposed method, solution of the given interval system of linear equation is $\underline{x}_1, \underline{x}_2$ =[8.75, -4.13] and $\overline{x}_1, \overline{x}_2$= [8.0909 , -4.4545] .

Example 4.3: The last example was taken from [11] and [9].

$$\begin{bmatrix}[2,3] & [0,1]\\ [1,2] & [2,3]\end{bmatrix}\begin{bmatrix}[\underline{x}_1, & \overline{x}_1]\\ [\underline{x}_2, & \overline{x}_2]\end{bmatrix} = \begin{bmatrix}[0, & 120]\\ [60, & 240]\end{bmatrix}$$

Applying the proposed method, solution of the given interval system of linear equation is $\underline{x}_1, \underline{x}_2$ =[0 ,17.1429] and $\overline{x}_1, \overline{x}_2$= [30 , 68.5714] .

5.0 DISCUSSION

Solutions of system discussed so far in example 4.1, 4.2 and 4.3 using existing and proposed methods are given in Table 1 below:

Problem	Existing Method		Proposed Method	
	Solution	Interval Width	Solution	Interval Width
Example 4.1	$\underline{x}_1, \overline{x}_1 = [0.6231, 1.2083]$ $\underline{x}_2, \overline{x}_2 = [-0.206, -0.052]$	[0.5852] [0.1555]	$\underline{x}_1, \overline{x}_1 = [0.7142, 1]$ $\underline{x}_2, \overline{x}_2 = [-0.1429, 0]$	[0.2858] [0.1429]
Example 4.2	$\underline{x}_1, \overline{x}_1 = [8.75, 8.0909]$ $\underline{x}_2, \overline{x}_2 = [-4.13, -4.4545]$	[0.6591] [0.3295]	$\underline{x}_1, \overline{x}_1 = [8.75, 8.0909]$ $\underline{x}_2, \overline{x}_2 = [-4.13, -4.4545]$	[0.6591] [0.3295]
Example 4.3	$\underline{x}_1, \overline{x}_1$ = [-27.406, 54.679] $\underline{x}_1, \overline{x}_1 = [-6.909, 110.546]$	[27.273] [103.637]	$\underline{x}_1, \overline{x}_1 = [0, 17.1429]$ $\underline{x}_2, \overline{x}_2 = [30, 68.5714]$	[17.1429] [38.5714]

From Table 1we can see that the results obtained in example 4.1 and 4.3 using the proposed method are better than the results obtained with the existence methods because the interval width is less than the interval width previously obtained by the existence methods. Also the results obtained in example 4.2 using the proposed method agrees with the result obtained in the existence method, though our methods is simple than the existence method.

5.1 CONCLUSION

In this paper, we proposed a method for solving interval systemof linear equation which is relatively new approach for solving this type of system of equations besides other approaches. It should be noted that in order to use the proposed method, the lower and upper bounds of the given matrix must be regular. Though the procedure is simple, but using the method we can easily find out a good solution with better result.

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