On the Uni-direction Model of Extreme Wave Profile for Freak Wave Events Induced by the Ocean Current.

[1]Ejinkonye I.O. and[2]Ifidon E.O.

[1]Department ofMathematics and Computer Science,Western Delta University, Oghara,Delta State, Nigeria.e-mail ejinkonye.ifeoma@yahoo.com Gsm +2348060297345

[2]Department of Mathematics, University of Benin, Benin City, Edo State, Nigeria.

Abstract

This study is essentially on the theory of ocean waveinduced by current in deep water. Both frequency, group and phase velocity dispersion curves were analyzed and displayed. Using non-dimensional wave parameters, it is deduced mathematically that the inducing ocean current and the interacting sea wave must be in opposition with regards to their respective phase speed. This result is in agreement with observation. The wave number at the point of blocking is calculated and deduced to be a function of the current speed.

A one–dimensional ray equations governing the evolution of wave packet interacting with ocean current in addition wasanalyzed. The solution confirms the existence of the focusing point earlier mentioned. The distance for which the monochromatic wave packet can penetrate into the current before being halted by opposing ocean current is also derived. At the blocking zone (caustic), it is proved in this study that, not only is the wave amplitude very large, the wave-length which is then proportional to the square of current velocity is similarly large. This is more pronounced in the event of strong ocean current.

Keywords; Rogue wave, Ocean current, Dispersion, Caustic, Wave packets.

1 Introduction

Observation of rogue wave events in some areas of intense ocean current is fairly regular. The most notable sea areas with such occurrences are those of Agulhas in South Africa and Gulf stream [1-7]. Lavrenov [1,2] observed an area in South Africa Agulhas current where current energy concentratessufficient enough to induce rogue wave event.

This study will regard this energy focusing area as a blocking zone. It has been generally considered, therefore, that wave/current interaction is an effective mechanism capable of initiating rogue wave event. A number of ocean going vessels, oil tankers and onshore marine structures had suffered the destructive effects in these sea areas arising from the activities of these giant sea monsters called rogue waves.

A number of theoretical papers in this consideration had been published on the evolution of rogue wave events. Kharif and Pelinovsky [8], gave an extensive review of the mathematical development in this consideration. In an attempt to establish the existence of extreme crest in wave profile, the distance between two adjacent orthogonal was considered. However, the theory applies to shallow water waves. It does not apply to unidirectional deep water sea waves in the current study in whichthe wave orthogonal are parallel.

There are some other theoretical papers on rogue wave events. Some of the more recent publications related to this study are those of [9-13].

The chapter seven in Introduction to Ocean Waves [14] seems quite revealing. It introduces the topic of rogue wave event in a systemic fashion and quite complete. However, it is assumed that the speed of the current opposing the wave motion is in opposition to that of the wave phase velocity. The constant of proportionality in this case is not assigned to its physical significance, so the derivations cannot be evaluated numerical.

Further, in the process of establishing the existence of exceptional large amplitude form of wave crest, the law for the conservation of wave action was invoked. However, from the kinematic equation for evolution of wave action, what is conserved turns to be the wave action flux. That is, wave action multiplied by the absolute velocity [14-16].

In addition, during the process of wave\current interaction, the same analysis effectively provides the optimum distance the wave packet is likely to penetrate into opposing current. However, of greater interest is the analysis of wave parameters where the dispersive blow-up (caustic or energy blocking zone) occurs.

The essence of this investigation is therefore to provide solutions for these mentioned lapses.Further, moredetailed description of wave parameters at the energy blocking zone will be suggested.

2 Ocean wave / current interaction

The average velocity of ocean current is no longer zero in this consideration but has velocity $\underline{U}(x,t)$. $\underline{U}$ is a slowly varying function of x and t but constant with depth profile. That is, $\underline{U}$ varies in horizontal direction essentially in a wave- length or period scale. The waves are conveyed by the moving current. The wave phase and group velocities are thus affected by the moving current. To a stationary observer, the absolute wave frequency ω is expressed by

$$\omega = \underline{K} \cdot \underline{U} + \Omega(\underline{K}H) \qquad (1)$$

At the ocean depth H, the wave relative frequency is $\Omega(KH)$,calculated from dispersion equation

$\Omega^2 = Kg\tanh(KH)$ (2)

g = acceleration of gravity

And $K = \sqrt{k^2 + l^2}$, $\underline{K} = \underline{i}k + \underline{j}l$, $\underline{U} = \underline{i}u + \underline{j}v$ [14]

Where $\underline{i}\ and\ \underline{j}$ are unit vectors in the direction of x and y rectangular coordinates respectively.

$\underline{K}.\underline{U}$ is effect of moving current on frequency and may be regarded as Doppler shift in optics. $\underline{K}.\underline{U}$ is positive or negative depending on whether the wave is propagating in same or opposite direction to that of the ocean current. Emphasis on this statement follows.

3 The significance of ocean current velocity $\underline{U}$

We introduce non-dimensional parameter $\delta = \frac{h_0}{\lambda_0}$ as the long wave parameter, where h_0 is a typical water depth below the undisturbed sea-surface, λ_0 is a typical wave length and hence, δ is a pure number. In this consideration the current is defined as the horizontal movement of the body of water. From the observation of the deep water long wave, the motionis essentially in x-direction. In this consideration, therefore only $x-$component of the motion will be analyzed.

Following [17]and [18], non-dimensional approach is, to some extent, adopted. In this regard, the wave dispersion equation in deep water is stated as (i) $\frac{kg}{\omega^2} = 1$ where all the quantities are in dimensional form (ii) $\frac{\delta_s \omega_s^2}{k_s} = 1$, where all the quantities are in non-dimensional form. From (ii) $\frac{\omega_s^2}{k_s^2} = c_s^2 = \frac{1}{k_s \delta_s}$ (subscript s indicates a pure number), $c = c_s\sqrt{gh_0}$, $k_s = \frac{1}{\delta_s c_s^2}$ and from equation (1), $\omega_s = k_s(c_s + u_s) = \frac{1}{\delta_s c_s^2}(c_s + u_s)$. Dropping the subscripts s then,

$$\omega c^2 \delta - c - u = 0. \quad (3)$$

Equation (3) determines the current controlled wave evolution.It will be applied to describe some related important physical events. The solution is in the form;

$$c = \frac{1}{2\delta\omega}\left(1 \pm \sqrt{1+4\delta\omega u}\right) = \frac{c_p}{2}\left(1 \pm \sqrt{1+4\delta\omega u}\right) \qquad (4)$$

$$c_p = \frac{1}{\delta\omega} \qquad (5)$$

Where c_p is wave-speed in absence of current. The solution $c = \frac{c_p}{2}\left(1 - \sqrt{1+4\delta\omega u}\right)$ is inapplicable for when $u = 0, c = 0$. That is, the wave cannot exist without current. This is not true. Water waves can involve with or without ocean current.

Thus, the more realistic solution is given by;

$$c = \frac{c_p}{2}\left(1 + \sqrt{1 + \frac{4u}{c_p}}\right) \qquad (6)$$

Equation (6) will apply being geophysical realistic because, when

$$u = 0, \quad c = c_p$$

Following [18], equation (6) leads to the following interesting results:

i. If u is in the same direction with c, than $c > c_p \ (u > 0)$.

ii. If u is in opposite direction to C, i.e. $u < 0$ and $c < c_p > 0$, the wave length shortens and the wave moves slower.

iii. If in addition $1 + \frac{4u}{c_p} < 0 \ \ i.e. \ u < -\frac{c_p}{4}$, c becomes complex. At this stage the wave will no longer propagate. What follows is equally quite interesting in this discussion.

In a steady state all wave parameters are time independent.Therefore, the wave action $\frac{E}{\Omega}$ is governed by the equation

$$\left(u + c_g\right)\frac{E}{\Omega} = \text{constant} = R_0 \qquad (7)$$

where R_0 is a constant. In deep water, $c_g = \frac{c}{2}$, E is the wave energy and is proportional to the square of wave amplitude A. Thus $\frac{E}{\Omega} = c_0 A^2, \ c_0 = \frac{\rho g}{2\pi\Omega}$. Equation (6) becomes

$$(u+\frac{c}{2})A^2 = \text{constant} \tag{8}$$

as $u \to \frac{-c_p}{4}$, $c \to \frac{-c_p}{2}$ from equation (6) i.e $u \to \frac{-c}{2}$ *and* $u+\frac{c}{2} \to 0$

Thus , $u+\frac{c}{2} \to 0$ and corresponds to $A \to \infty$ for equation (8) to apply . It is interesting to derive the phenomena of caustic in this study. The importance of this result in the study of physical optics and related phenomena (caustic) is of note. The existence of energy blocking (caustic) appears in this present study and relates this investigation to the occurrence of extreme crest and rogue wave phenomenon.

This simple analytical model depicts the existence of large crest in the wave profile. This was based on the adjustment between wave relative velocity and that of opposing ocean current. As already mentioned, the conservation of wave action flow so involved has wave absolute frequency as the numerator and its variation should be involved in any improve complete model. A more complete form related to wave action evolution will therefore be revisited in the subsequentsection of this study.

4 The dispersion relations for deep water wave evolution

The consideration will concern the deep water wave dispersion relations. These relations involve the dependence of some wave parameters such as frequency, group and phase velocity on wave number correspondingly. In the case of the wave\ocean current interactions, these parameters will refer to their absolute values. The frequency, group and phase velocity dispersion equations are now analyzed respectively in the presence of moving ocean current u. Negative u implies that the wave motion is opposed by that of ocean current.

4.1 The frequency dispersion curve.

The equation that governs the dependence of the wave absolute frequency ω on the wave number k related to it in deep water is as follows

$$\omega = \sqrt{kg} - ku \tag{9}$$

In equation (9) and subsequent equations, the current opposes the wave motion.

ω=0, gives $k = \left(0, \frac{g}{u^2}\right)$. Hence, on positive k-axis irrespective of the sign of u i.e. direction of the current. The extreme values of $\omega(k)$ is given by the value k such that $\omega'(k) = 0$. That is

$$k = \frac{g}{4u^2} > 0 \tag{10}$$

$\omega''(k)$ provides the nature of the extreme value of $\omega(k)$. Thus

$$\omega''\left(\frac{g}{4u^2}\right) = \frac{-2u^3}{g} < 0 \tag{11}$$

Equation (11) suggests that $\omega(k)$ has maximum value at $k = \frac{g}{4u^2}$

$$\omega\left(\frac{g}{4u^2}\right) = \frac{g}{4u} \tag{12}$$

Then, from equation (11), $\omega(k)$ has relative (absolute) maximum value. Thus, the extreme value of $\omega(k_0) = \frac{g}{4u}$ is provide by $k_0 = \frac{g}{4u^2}$. It is thus the peak frequency at the caustic or blocking zone.

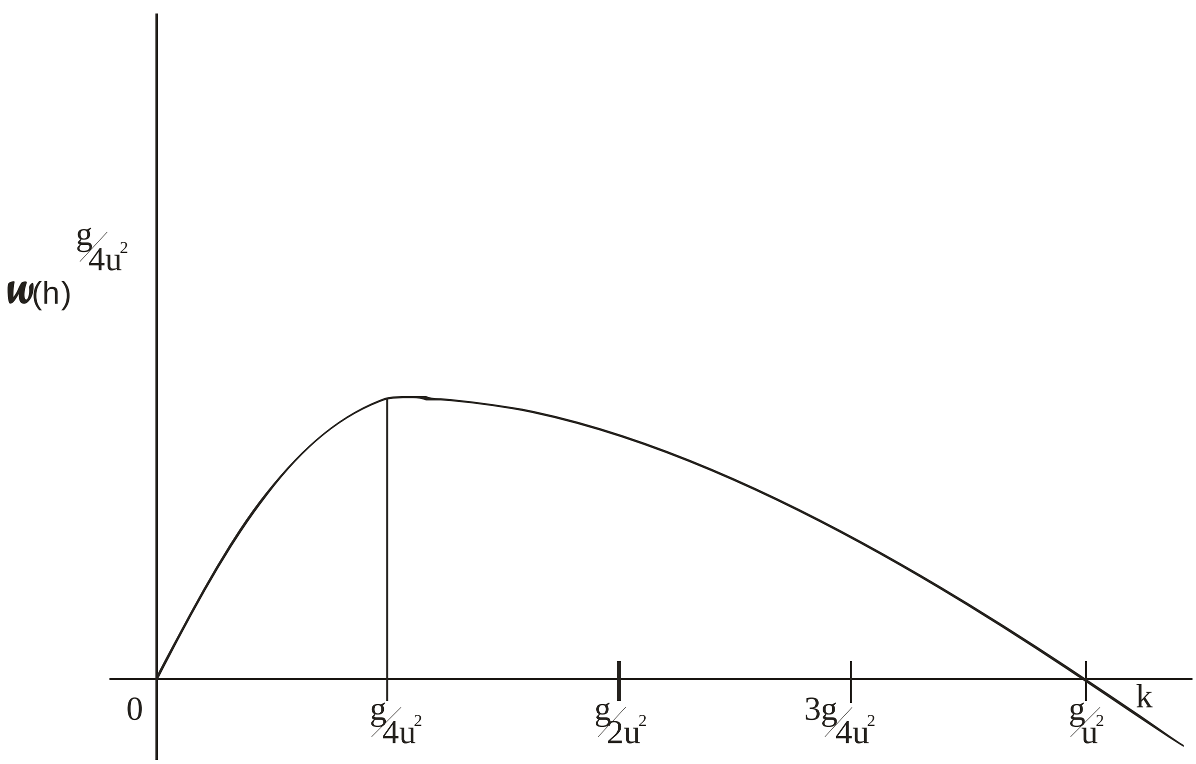

Fig 1: Frequency/Wave Number Dispersion Curve

4.2 The group velocity dispersion curve

The curve fig. (2) is expressed by the equation

$$c_g = \frac{1}{2}\sqrt{\frac{g}{k}} - u \tag{13}$$

Equation (13) has no extreme value but it crosses k-axis where $c_g = 0$ corresponding to $k = \frac{g}{4u^2}$. Interestingly, this is the value of k for which $\omega(k)$ is extreme. This development appears to have explained the physical definition of caustic or energy blocking in wave number coordinate. It is, thus, suggested that notonly that the absolute group velocity vanishes but also, the absolute wave frequency becomes extreme at caustic.

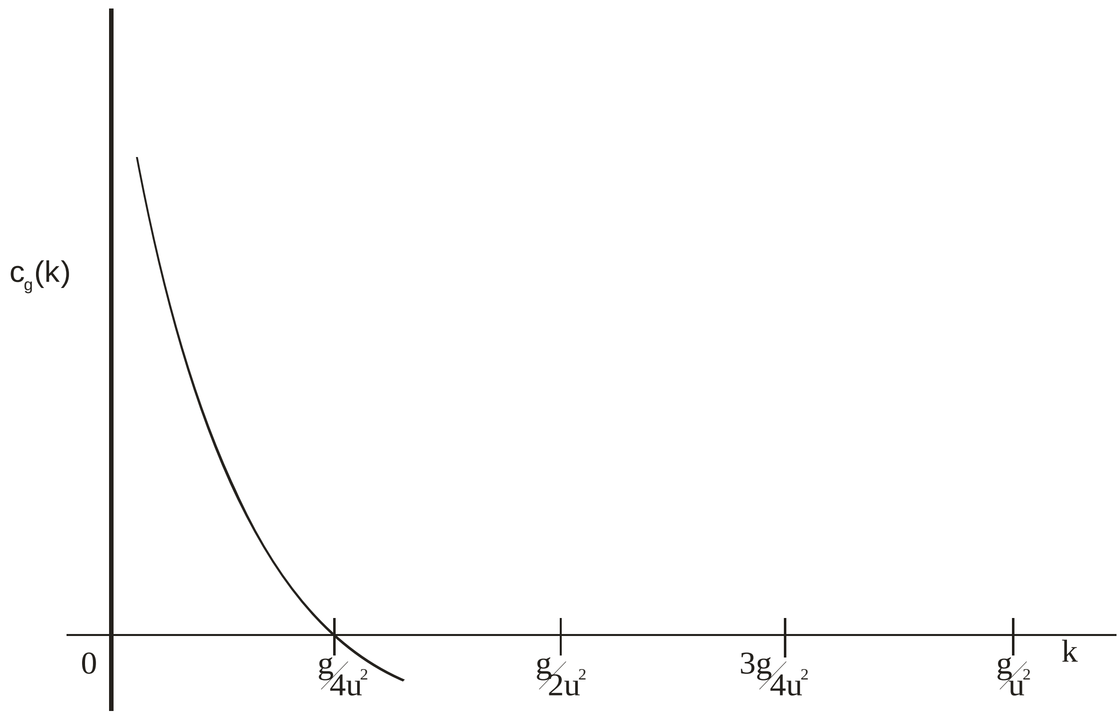

Fig 2: The Group Velocity/Wave Number Dispersion Curve

4.3 The phase velocity dispersion curve

The wave number and phase velocity relation fig. 3is provided in deep water by the equation.

$$c = \sqrt{\frac{g}{k}} - u, \qquad (14)$$

$$c = 0 \text{ gives } k = \frac{g}{u^2} \qquad (15).$$

This corresponds to the value of k for which $\omega(k) = 0$. At the caustic, $k = \frac{g}{4u^2}$, the corresponding phase velocity $c = 3u$ if $u > 0$ and $c = u$ if $u < 0$;not zero from equation (14)

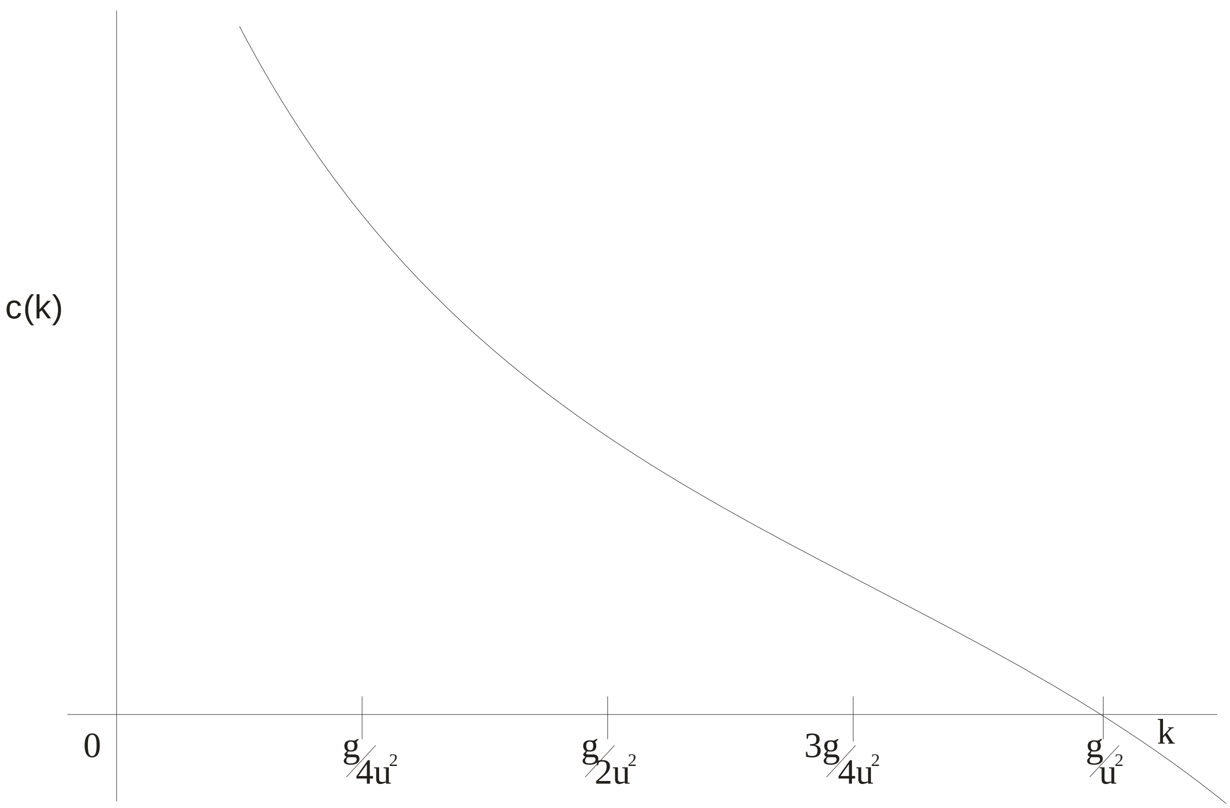

Fig 3: The Phase Velocity/Wave Number Dispersion Curve

The distance from the initial position of the wave packet to its maximum penetration had been suggested by [14]. This study further suggests that wave packet crosses the energy blocking zone (caustic) with absolute speed c which is three time that of ocean current. if $u > 0$.It is noted that c is in ms^{-1} whilst u is in $cm\ s^{-1}$, thus, u is relatively quite small. The simple approach adopted in this study appears to have lead, in almost trivial way to what appear to be, not only interesting, but important physical result in global geophysics.

5 Application of ray equations for uni-directional wave motion

In this consideration, the x-axis is perpendicular to the wave front and the entire motion will be uniform with y-axis; t>0 is the time. These assumptions are reasonably in agreement with the observed evolutions of rogue wave event following [15-16]. The current speed $u(x)$ is taken as

$$u(x) = -\omega_0 x, \quad x > 0 \tag{16}$$

$$\omega_0 = \frac{2\pi}{T_0}, T_0 = 12s \ then \ \omega_0 = 0.524 rad/s$$

where ω_0 is the peak frequency in the model of frequency narrow banded spectrum. The group velocity c_g in deep water is defined by

$$c_g = \frac{\partial \Omega}{\partial k} = \frac{1}{2}\sqrt{\frac{g}{k}}, \quad \Omega = \sqrt{kg} \tag{17}$$

Thus, the refraction equations governing uni-directional motion in deep water are [15].

$$\frac{dx}{dt} = -\omega_0 x + \frac{1}{2}\sqrt{\frac{g}{k}} \tag{18}$$

$$\frac{dk}{dt} = k\omega_0 \tag{19}$$

The ray equation (18) and (19) govern the evolution of a wave packet with carrier wave number k(t) and located at $x(t)$ (19) has solution of the form

$$k = k_0 e^{\omega_0 t} \tag{20a}$$

$$where \quad k_0 = k(0) \tag{20b}$$

Equation (20b) is initial data for wave-number evolution. It will be calculated in subsequent derivations. The evolutional form of wave-number k(t) is depicted in equation (20a).Equation (18) using equation (20a) reduces to

$$\frac{dx}{dt}+\omega_0 x=\frac{1}{2}\sqrt{\frac{g}{k_0}}e^{-\delta_1 t} \qquad (21)$$

$$\delta_1=\frac{\omega_0}{2} \quad ie \quad \omega_0=2\delta_1 \qquad (22)$$

Equation (21) has solution of the form

$$xe^{2\delta_1 t}=\frac{1}{2\delta_1}\sqrt{\frac{g}{k_0}}e^{\delta_1 t}+A$$

$$x=\frac{1}{2\delta_1}\sqrt{\frac{g}{k_0}}e^{-\delta_1 t}+Ae^{-2\delta_1 t}$$

$$x(0)=0 \qquad ie \qquad A=-\frac{1}{2\delta_1}\sqrt{\frac{g}{k_0}}$$

Thus, $$x=\frac{1}{2\delta_1}\sqrt{\frac{g}{k_0}}\left(e^{-\delta_1 t}-e^{-2\delta_1 t}\right) \qquad (23)$$

Then, as $t\to\infty,\ x\to 0,\ k\to\infty$ (23) defines the location of the wave-packet at time t.

The time (t_0) for the optimal location of wave package (ie stationary value of $x(t)$) is provided by $\frac{dx(t_0)}{dt}=0$ corresponding to $c_g=0$

That is, $-e^{-\delta_1 t_0}+2e^{-2\delta_1 t_0}=0 \qquad or \quad e^{\delta_1 t_0}=2$

$$t_0=\frac{\ln 2}{\delta_1}=\frac{\ln 4}{\omega_0}$$

$x_0=x\left(\frac{2\ln 2}{\omega_0}\right)=x\left(\frac{\ln 4}{\omega_0}\right)$ = location of the optimal position.

$$= \frac{1}{\omega_0}\sqrt{\frac{g}{k_0}}\left[\frac{1}{2}-\frac{1}{4}\right] = \frac{1}{4\omega_0}\sqrt{\frac{g}{k_0}}$$

$$i.e \quad x_0 = \frac{1}{4\omega_0}\sqrt{\frac{g}{k_0}} \tag{24}$$

The corresponding speed of the ocean currentis given by

$$u = \frac{1}{4}\sqrt{\frac{g}{k_0}} = \omega_0 x_0 \tag{25}$$

equation (25) suggests that this is about a fourth of wave speed $\sqrt{\frac{g}{k_0}}$. The corresponding wave number is given by

$$k(t_0) = k_0 e^{\ln 4} = 4k_0 \tag{26}$$

Thus, at the point of maximum penetration of the wave packet, the original wave number is increased by four fold, the location of wave packet and current speed are calculated from (24) and (25).

5.1 Wave packet penetration

The parameters for the wave packet penetrating in the energy blocking zone are here re-considered. The wave number k_0 at the blocking zone is given by $k_0 = \frac{g}{4u^2}$ with corresponding frequency $\omega(k_0) = \frac{3g}{4u}$. The time t_0 for the optimum penetration into the blocking zone (caustic) is given by $t_0 = \frac{2\ln 2}{\omega_0}$. For this value of t_0, $k_0 = 4k(0)$ where $k(0)$ is the initial data for wave number. $k(0) = \frac{g}{u^2}$ and corresponding wave length $L_0 = \frac{2\pi u^2}{g}$. The distance for which the wave packet penetrates is $x(t_0)$. After this distance, the wave packet appears to have lost its inherent energy whilst crossing the blocking zone. Thus,

$$x(t_0) = \frac{4u}{\omega_0}\left[\exp\left(\frac{-\ln 4}{2}\right) - \exp(-\ln 4)\right] = \frac{4u}{\omega_0}\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{u}{\omega_0}$$

i.e. this result verifies our earlier assumption. Thus $x_0 = x(t_0)$ is obtained as a function of current velocity in final analysis.

Table 1

u(cm/s)	L_0(cm)	$x(t_0)$(cm)
10	0.641	19.084
12	0.924	22.901
14	1.257	26.718
16	1.642	30.534
18	2.078	34.351
20	2.566	38.168

Concluding from table 1, the wave length diminishes from order of hundreds of meter to order of centimeter in the blocking zone. What is the physics behind this scenario, from deep water dispersion relation, relative group velocity $c_g = \frac{1}{2}\sqrt{\frac{gl_0}{2\pi}}$, hence as c_g becomes small on approaching the blocking zone and wave length shrinks

6 Development of large crest height

Using equations (13) and the representation of (17) in deep water,

$$\frac{u + c_g}{\omega} = \frac{u + \frac{1}{2}\sqrt{\frac{g}{k}}}{\sqrt{gk} + ku} = \frac{2u\sqrt{k} + \sqrt{g}}{2k\left(u\sqrt{k} + \sqrt{g}\right)} \tag{27}$$

Thus,

$$\lim_{k\to\infty}\left[\frac{2u\sqrt{k} + \sqrt{g}}{2k\left(u\sqrt{k} + \sqrt{g}\right)}\right] = \lim_{k\to\infty}\left[\frac{u}{\sqrt{k}\left[2\left(u\sqrt{k} + \sqrt{g}\right) + u\sqrt{k}\right]}\right]$$

$$\lim_{k\to\infty}\left[\frac{u}{\sqrt{k}\left[3u\sqrt{k} + 2\sqrt{g}\right]}\right] \to 0$$

But from (1) and (13)

$$\lim_{k\to\infty}\frac{\left(u + \frac{1}{2}\sqrt{\frac{g}{k}}\right)}{ku + \sqrt{kg}}.A^2 = \text{constant} \tag{28}$$

Hence, A^2 will grow very large to satisfy equation (28). This is the verification of earlier conclusion. The present analytical study therefore, clearly suggests that wave current focusing can induce ocean wave mode with exceptionally high crest.

7 Effectiveness of wave/current energy focusing

This mechanism requires the following for its effectiveness

1. Atmospheric wind flow: this then generates progressive sea wave which moves in the direction corresponding to that of the wind.
2. Opposing the motion is the steady moving ocean current [8,16]. Of note are ocean currents such as those of Agulhas in South Africa and Gulf stream of North America as was observed by [1,2]. Waves with extreme high crests often appear in these ocean areas. Encounter between these large amplitude waves and ocean vessels (commercial and military) are occasionally observed in these localities. The relationship between the opposing ocean waves, ocean current and the resultant extreme high crest is thus suggested in this investigation.
3. It is emphasized that the totality of energy in this consideration is derived from opposing ocean current. This energy source generates ocean wave with extreme crest followed deep trough and so, it goes.
   It may be mentioned that at blocking zone where the group velocity vanishes, the wave amplitude tends to infinity. In classical physics, this is a catastrophic phenomenon. However, observational evidence [1] clearly indicates that the wave amplitude, though large, is still bounded and measurable. It follows that immediately the wave height is about tending to breaking amplitude, the wave growth is halted by the inherent non-linearity and dispersion associated with the stage of its development.

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