PIECEWISE CONTINUOUS TRIAL FUNCTIONS IN THE FINITE ELEMENT SOLUTION OF ONE DIMENSIONAL FIELD PROBLEM USING RAYLEIGH-RITZ METHOD

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Abstract

One the flaws of the traditional variational methods is that the trial functions are arbitrarily chosen and the weighted integrals are applied globally over the entire region of interest. Consequently for complex regions , the boundary conditions as well as the physics of the problem are not satisfied.

In this paper, we present the finite element method, it is an element wise application of the Rayleigh-Ritz method. its essence is the minimization of an appropriate functional, which is developed on adoption of the Euler –Lagrange's equation. The discretization of the region of interest is done using linear elements permitting a close approximation at discrete nodes. The element functional minimization results in a series of algebraic equations which on assembly using the direct stiffness method yields the system equation. The required nodal degree of freedom is obtained after imposing the boundary conditions

1.0Introduction

The finite element method [1] is one of the foremost engineering tools for the approximation of differential equations .It is a technique for constructing approximation functions required in an element-wise application of the Rayleigh-Ritz[2]. Obviously, most physical phenomena are modeled through the mathematical formulation of the physical process as well as the numerical analysis of the mathematical model. The mathematical formulation of a physical process results in differential equations while the derivation of the governing equations for most problems is not difficult, the solution by exact methods of analysis is aformidable task. In such cases, approximate method of analysis provides alternative means of finding solutions. Several approximate methods for solving both initial and boundary value problems exist in literature amongst these include the Euler method [3], the weighted residual method [4] e.t.c)

1.1 Formulation of the Euler-Lagrange's equation

The necessary condition for the existence of an extremum of the functional

$$I = \int_a^b F(x, y, y_x)\, dx \tag{1.0}$$

is that its first variation δI must be zero, or

$$\delta I = \delta \int_a^b F(x, y, y_x) = 0 \tag{1.1}$$

Provided that

$$\frac{\partial F}{\partial y_x} \delta y \bigg|_a^b = 0 \tag{1.2}$$

is satisfied[5,6]

With brevity in calculus of variation, from eqn.(1.0) we can derive the Euler-Lagrange equation using the definition of variation as well as integration by parts to have

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_x}\right) = 0 \, (1.3)$$

Eq.(1.3) is referred to as the Euler-Lagrange equation. It is a differential equation with y and y_x as pseudo-independent variables.

Example 1

Solve the differential equation

$$\frac{d^2T}{dx^2}+1000x^2=0 \qquad 0\le x\le 1 \tag{1.4}$$

$$T(0)=T(1)=0 \tag{1.5}$$

Using Rayleigh-Ritz method

The exact solution is

$$T(x)=\frac{1000}{12}x\left(1-x^3\right) \tag{1.6}$$

Solution

Now in terms of the problem at hand, the Euler- Lagrange equation is

$$\frac{\partial F}{\partial T}-\frac{d}{dx}\left(\frac{\partial F}{\partial T_x}\right)=0 \tag{1.7}$$

and

$$\left.\frac{\partial F}{\partial T_x}\delta T\right|_0^1=0 \tag{1.8}$$

Since the value of T is to be held fixed at either end of the interval, $0\le x\le 1$,the variation of T or δT must be zero at these two points . Therefore, Eq.(1.8) is satisfied by virtue of the conditions $T\left(0\right)=0$ and $T\left(1\right)=0$, since $\delta T=0$ at these points. Let us now turn to Eq.(1.7) and make a term-by-term comparison with the governing equation expressed in (1.4), which yields

$$\frac{\partial F}{\partial T}=1000x^2 \tag{1.9}$$

and

$$\frac{d^2T}{dx^2} = -\frac{d}{dx}\left(-\frac{dT}{dx}\right) = -\frac{d}{dx}\left(\frac{\partial F}{\partial T_x}\right) \quad (1.10)$$

Which implies that

$$\frac{\partial F}{\partial T_x} = -\frac{dT}{dx} = -T_x \quad (1.11)$$

Hence

$$\frac{\partial F}{\partial T_x} = -T_x \qquad (1.12)$$

From (1.9) , we have

$$F = 1000x^2T + f\left(T_x\right) \qquad (1.13)$$

And from Eq.(1.12) implies

$$F = -\frac{1}{2}T_x^{\ 2} + g\left(T\right) \qquad (1.14)$$

By direct substitutions and further comparison of Eq.(1.13) and with Eq.(1.14) yields

$$f\left(T_x\right) = -\frac{1}{2}T_x^{\ 2} \qquad (1.15)$$

and

$$g\left(T\right) = 1000x^2T \qquad (1.16)$$

So fromequations (1.13),(1.15) the functional I is given by

$$I = \int_0^1 \left[1000x^2T - \frac{1}{2}\left(\frac{dT}{dx}\right)^2\right]dx \qquad (1.17)$$

We assume a two-term approximate solution as

$$\tilde{T}(x) = a_1 N_1(x) + a_2 N_2(x) = a_1 x\left(1 - x^2\right) + a_2 x\left(1 - x^4\right) \quad (1.18)$$

$$\frac{d\tilde{T}}{dx} = a_1\left(1 - 3x^2\right) + a_2\left(1 - 5x^4\right) \quad (1.19)$$

Substituting Eq.(1.19) into Eq.(1.17), we have

$$\mathrm{I} = \int_0^1 \left[1000x^2 \left\{ a_1\left(x - x^3\right) + a_2\left(x - x^5\right) \right\} - \frac{1}{2} \left\{ a_1\left(1 - 3x^2\right) + a_2\left(1 - 5x^4\right) \right\}^2 \right] dx \quad (1.20)$$

Expanding we have,

$$\mathrm{I} = \int_0^1 \left[1000 a_1 x^3 - 1000 a_1 x^5 + 1000 a_2 x^3 - 1000 a_2 x^7 \right]$$

$$-\frac{1}{2}\left[a_1^2\left(1 - 6x^2 + 9x^4\right) + 2a_1 a_2\left(1 - 3x^2 - 5x^4 + 15x^6\right) + a_2^2\left(1 - 10x^4 + 25x^8\right) \right] dx \quad (1.21)$$

Integrating with respect to x, we have

$$\left[\frac{1000}{4} a_1 x^4 - \frac{1000}{6} a_1 x^6 + \frac{1000}{4} a_2 x^4 - \frac{1000}{8} a_2 x^8 \right]_0^1$$

$$-\frac{1}{2}\left[a_1^2\left(x - 2x^3 + \frac{9}{5}x^5 \right) + 2a_1 a_2\left(x - x^3 - x^5 + \frac{15}{7}x^7 \right) + a_2^2\left(x - 2x^5 + \frac{25}{9}x^9 \right) \right]_0^1 \quad (1.22)$$

Evaluating further, we have

$$\mathrm{I} = \frac{1000}{4} a_1 - \frac{1000}{6} a_1 + \frac{1000}{4} a_2 - \frac{1000}{8} a_2 - \frac{1}{2}\left[a_1^2\left(1 - 2 + \frac{9}{5}\right) + 2a_1 a_2\left(1 - 1 - 1 + \frac{15}{7}\right) + a_2^2\left(1 - 2 + \frac{25}{9}\right) \right]$$

$$= a_1\left(\frac{1000}{4} - \frac{1000}{6} \right) + a_2\left(\frac{1000}{4} - \frac{1000}{8} \right) - \frac{1}{2}\left[a_1^2 \frac{4}{5} + 2a_1 a_2 \frac{8}{7} + a_2^2 \frac{16}{9} \right]$$

$$= \frac{1000}{12} a_1 + \frac{1000}{8} a_2 - \frac{2}{5} a_1^2 - \frac{8}{7} a_1 a_2 - \frac{8}{9} a_2^2 \quad (1.23)$$

We now extremize the functional I and set it equal to zero

$$\frac{\partial \text{I}}{\partial a_1} = \frac{1000}{12} - \frac{4}{5}a_1 - \frac{8}{7}a_2 = 0 \tag{1.24}$$

$$\frac{4}{5}a_1 + \frac{8}{7}a_2 = \frac{1000}{12} \tag{1.25}$$

$$\frac{\partial \text{I}}{\partial a_2} = \frac{1000}{8} - \frac{8}{7}a_1 - \frac{16}{9}a_2 = 0 \tag{1.26}$$

$$\frac{8}{7}a_1 + \frac{16}{9}a_2 = \frac{1000}{8} \tag{1.27}$$

Solving equations Eqs. (1.25) and (1.27), we have

$$a_1 = 48.9647, \qquad a_2 = 38.6336 \tag{1.28}$$

Hence the approximate solution given by Rayleigh-Ritz method is

$$T_{RR}(x) = 48.9647x\left(1 - x^2\right) + 38.6336x\left(1 - x^4\right) \tag{1.29}$$

1.3 Finite element method

Example 2

In this section, we apply the procedures for finite element method to resolve the problem posed in example 1

Solution

We first discretize the domain into five elements of six equally spaced node points.

From example 1, **the functional to the differential equation is**

$$\text{I} = \sum_{e=1}^{5} \left\{ \frac{dT}{dx} T^e \bigg|_{x_i}^{x_j} + \int_{x_i}^{x_j} \left[1000x^2 T^e - \frac{1}{2}\left(\frac{dT^e}{dx}\right)^2 \right] dx \right\} \tag{1.30}$$

Let us drop the summation and write the functional for a typical element e as

$$I^{(e)} = \frac{dT}{dx}T^e \Big|_{x_i}^{x_j} + \int_{x_i}^{x_j} \left[1000x^2 T^e - \frac{1}{2}\left(\frac{dT^e}{dx}\right)^2 \right] dx \tag{1.31}$$

Where

$$I = \sum_{e=1}^{5} I^e \tag{1.32}$$

This summation is really considered at the assemblage step later.

Recall that

$$\tilde{T}^e = a_1 N_1 + a_2 N_2 = [N] a^e \tag{1.33}$$

Where the piecewise continuous trial function or shape function is given by

$$[N] = [N_1 \quad N_2] \quad , \qquad a = [a_1 \quad a_2]^T \tag{1.34}$$

Now

$$\frac{\partial (Na^e)}{\partial a^e} = \frac{\partial T}{\partial a} \equiv \begin{bmatrix} \frac{\partial T}{\partial a_1} \\ \frac{\partial T}{\partial a_2} \end{bmatrix} = \begin{bmatrix} N_1(x) \\ \\ N_2(x) \end{bmatrix} = N^T \tag{1.35}$$

we now extremize the functional with respect to the vector a^e to have

$$\frac{\partial I^e}{\partial a^e} = \frac{dT}{dx}\frac{\partial T^e}{\partial a^e}\Big|_{x_i}^{x_j} + \int_{x_i}^{x_j} \left[1000x^2 \frac{\partial T^e}{\partial a^e} - \frac{1}{2}\frac{\partial}{\partial a^e}\left(\frac{\partial T^e}{\partial x}\right)^2 \right] dx = 0 \tag{1.36}$$

The second term under the integral can be evaluated thus

$$\frac{1}{2}\frac{\partial}{\partial a^e}\left(\frac{\partial T^e}{\partial x}\right)^2 = \frac{1}{2}(2)\frac{\partial T^e}{\partial x}\frac{\partial}{\partial a^e}\left(\frac{\partial T^e}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial T^e}{\partial a^e}\right)\frac{\partial T^e}{\partial x} = \frac{\partial N^T}{\partial x}\frac{\partial N}{\partial x}a^e \tag{1.37}$$

Substituting equations (1.35) and (1.37) into equation (1.36), we have

$$\frac{dT}{dx}N^T\bigg|_{x_i}^{x_j} + \int_{x_i}^{x_j} 1000x^2 N^T dx - \int_{x_i}^{x_j} \frac{\partial N^T}{\partial x}\frac{\partial N}{\partial x} a^e dx = 0 \tag{1.38}$$

$$\left[\int_{x_i}^{x_j} \frac{\partial N^T}{\partial x}\frac{\partial N}{\partial x} dx\right] a^e = \frac{dT}{dx}N^T\bigg|_{x_i}^{x_j} + \int_{x_i}^{x_j} 1000x^2 N^T dx \tag{1.39}$$

Which is of the form

$$K^e a^e = f^e \tag{1.40}$$

Where

$$K^e = \int_{x_i}^{x_j} \frac{\partial N^T}{\partial x}\frac{\partial N}{\partial x} dx \tag{1.41}$$

and

$$f^e = \frac{dT}{dx}N^T\bigg|_{x_i}^{x_j} + \int_{x_i}^{x_j} 1000x^2 N^T dx \tag{1.42}$$

Next we evaluate Eqs (1.41) and (1.42)

Recall that

$$\frac{\partial\left(N^T\right)}{\partial x} = \frac{\partial}{\partial x}\begin{bmatrix} N_i(x) \\ N_j(x) \end{bmatrix} = \frac{\partial}{\partial x}\begin{bmatrix} \frac{x_j - x}{L} \\ \frac{x - x_i}{L} \end{bmatrix} = \begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} \tag{1.43}$$

$$\frac{\partial(N)}{\partial x} = \frac{\partial}{\partial x}\begin{bmatrix} N_i(x) & N_j(x) \end{bmatrix} = \frac{\partial}{\partial x}\begin{bmatrix} \frac{x_j - x}{L} & \frac{x - x_i}{L} \end{bmatrix} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \tag{1.44}$$

Thus $K^e = \int_{x_i}^{x_j} \frac{\partial N^T}{\partial x}\frac{\partial N}{\partial x} dx = \int_{x_i}^{x_j} \begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix}\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} dx = \begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix}\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\left(x_j - x_i\right)$ (1.45)

But $x_j - x_i = L$

Hence

$$K^e = \frac{1}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{1.46}$$

The integrated term in Eq.(1.38) is evaluated as follows

$$\left.\frac{dT}{dx}N^T\right|_{x_i}^{x_j} = \frac{dT}{dx}(x_j)N^T(x_j) - \frac{dT}{dx}(x_i)N^T(x_i) \tag{1.47}$$

But from the characteristics of the shape functions

$$\begin{aligned} N_i &= 1 \; at \;\; x = x_i \;\; and \;\; 0 \;\; at \;\; x = x_j \\ N_j &= 1 \; at \;\; x = x_j \;\; and \;\; 0 \;\; at \;\; x = x_i \end{aligned} \tag{1.48}$$

$$N^T(x_j) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad and \quad N^T(x_i) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{1.49}$$

Therefore, the integrated term becomes

$$\left.\frac{dT}{dx}N^T\right|_{x_i}^{x_j} = \frac{dT}{dx}(x_j)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{dT}{dx}(x_i)\begin{bmatrix} -1 \\ 0 \end{bmatrix} \tag{1.50}$$

We then evaluate $\int_{x_i}^{x_j} 1000x^2 N^T dx$,

$$\int_{x_i}^{x_j} 1000x^2 N^T dx = \int_{x_i}^{x_j} 1000x^2 \begin{bmatrix} \dfrac{x_j - x}{x_j - x_i} \\ \dfrac{x - x_i}{x_j - x_i} \end{bmatrix} dx = \frac{1000}{x_j - x_i} \int_{x_i}^{x_j} \begin{bmatrix} x^2 x_j - x^3 \\ x^3 - x^2 x_i \end{bmatrix} dx$$

$$= \frac{1000}{x_j - x_i} \begin{bmatrix} \dfrac{x^3}{3} x_j - \dfrac{x^4}{4} \\ \dfrac{x^4}{4} - \dfrac{x^3}{3} x_i \end{bmatrix} \Bigg|_{x_i}^{x_j} = \frac{1000}{x_j - x_i} \begin{bmatrix} \dfrac{x_j^4 - 4x_i^3 x_j + 3x_i^4}{12} \\ \\ \dfrac{3x_j^4 - 4x_j^3 x_i + x_i^4}{12} \end{bmatrix} \quad (1.51)$$

$$= \frac{1000}{12} \frac{1}{x_j - x_i} \begin{bmatrix} x_j^4 - 4x_i^3 x_j + 3x_i^4 \\ \\ 3x_j^4 - 4x_j^3 x_i + x_i^4 \end{bmatrix}$$

Since our domain is discretized into five elements, each element size $x_j - x_i = 0.2$

Hence for element one

$$\begin{matrix} & & & & 1 & 2 & \\ K^{(1)} = \dfrac{1}{x_2 - x_1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \dfrac{1}{0.2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = & \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} & \begin{matrix} 1 \\ 2 \end{matrix} \end{matrix} \quad (1.52)$$

The node numbers indicate the rows and columns of $[K]$ *and* $\{f\}$

For element two

$$\begin{matrix} & & & & 2 & 3 & \\ K^{(2)} = \dfrac{1}{x_3 - x_2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \dfrac{1}{0.2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = & \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} & \begin{matrix} 2 \\ 3 \end{matrix} \end{matrix} \quad (1.53)$$

For element three

$$\begin{matrix} & & & & & & 3 & 4 & \end{matrix}$$

$$K^{(3)} = \frac{1}{x_4 - x_3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{0.2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \begin{matrix} 3 \\ 4 \end{matrix} \tag{1.54}$$

For element four

$$\begin{matrix} & & & & & & 4 & 5 & \end{matrix}$$

$$K^{(4)} = \frac{1}{x_4 - x_3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{0.2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \begin{matrix} 4 \\ 5 \end{matrix} \tag{1.55}$$

For element five

$$\begin{matrix} & & & & & & 5 & 6 & \end{matrix}$$

$$K^{(5)} = \frac{1}{x_4 - x_3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{0.2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \begin{matrix} 5 \\ 6 \end{matrix} \tag{1.56}$$

For $f^{(1)}$,we have

$$f^{(1)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{dT(x_2)}{dx} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\frac{dT(x_1)}{dx} + \frac{1000}{(12)(0.2)}\begin{bmatrix} (0.2)^4 - 4(0.2)(0.0)^3 + 3(0.0)^4 \\ 3(0.2)^4 - 4(0.0)(0.2)^3 + (0.0)^4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{dT(x_2)}{dx} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\frac{dT(x_1)}{dx} + \begin{bmatrix} 0.667 \\ 2.000 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix} \tag{1.57}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{dT(x_2)}{dx} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\frac{dT(x_1)}{dx} + \begin{bmatrix} 0.667 \\ 2.000 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{dT(x_3)}{dx} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\frac{dT(x_2)}{dx} + \frac{1000}{(12)(0.2)}\begin{bmatrix} 0.0256 - 0.0128 + 0.0048 \\ 0.0768 - 0.0512 + 0.0016 \end{bmatrix}$$

$$=\begin{bmatrix}0\\1\end{bmatrix}\frac{dT(x_3)}{dx}+\begin{bmatrix}-1\\0\end{bmatrix}\frac{dT(x_2)}{dx}+\begin{bmatrix}7.3333\\11.3333\end{bmatrix}\begin{matrix}2\\3\end{matrix} \tag{1.58}$$

$$f^{(3)}=\begin{bmatrix}0\\1\end{bmatrix}\frac{dT(x_4)}{dx}+\begin{bmatrix}-1\\0\end{bmatrix}\frac{dT(x_3)}{dx}+\frac{1000}{(12)(0.2)}\begin{bmatrix}(0.6)^4-4(0.6)(0.4)^3+3(0.4)^4\\3(0.6)^4-4(0.4)(0.6)^3+(0.4)^4\end{bmatrix}$$

$$=\begin{bmatrix}0\\1\end{bmatrix}\frac{dT(x_4)}{dx}+\begin{bmatrix}-1\\0\end{bmatrix}\frac{dT(x_3)}{dx}+\frac{1000}{(12)(0.2)}\begin{bmatrix}0.1296-0.1536+0.0768\\0.3888-0.3456+0.0256\end{bmatrix}$$

$$=\begin{bmatrix}0\\1\end{bmatrix}\frac{dT(x_4)}{dx}+\begin{bmatrix}-1\\0\end{bmatrix}\frac{dT(x_3)}{dx}+\begin{bmatrix}21.6667\\28.6667\end{bmatrix}\begin{matrix}3\\4\end{matrix} \tag{1.59}$$

$$f^{(4)}=\begin{bmatrix}0\\1\end{bmatrix}\frac{dT(x_5)}{dx}+\begin{bmatrix}-1\\0\end{bmatrix}\frac{dT(x_4)}{dx}+\frac{1000}{(12)(0.2)}\begin{bmatrix}(0.8)^4-4(0.8)(0.6)^3+3(0.6)^4\\3(0.8)^4-4(0.6)(0.8)^3+(0.6)^4\end{bmatrix}$$

$$=\begin{bmatrix}0\\1\end{bmatrix}\frac{dT(x_5)}{dx}+\begin{bmatrix}-1\\0\end{bmatrix}\frac{dT(x_4)}{dx}+\frac{1000}{(12)(0.2)}\begin{bmatrix}0.4096-0.6912+0.3888\\1.2288-1.2288+0.1296\end{bmatrix}$$

$$=\begin{bmatrix}0\\1\end{bmatrix}\frac{dT(x_5)}{dx}+\begin{bmatrix}-1\\0\end{bmatrix}\frac{dT(x_4)}{dx}+\begin{bmatrix}44.6667\\54.0000\end{bmatrix}\begin{matrix}4\\5\end{matrix} \tag{1.60}$$

$$f^{(5)}=\begin{bmatrix}0\\1\end{bmatrix}\frac{dT(x_6)}{dx}+\begin{bmatrix}-1\\0\end{bmatrix}\frac{dT(x_5)}{dx}+\frac{1000}{(12)(0.2)}\begin{bmatrix}(1.0)^4-4(1.0)(0.8)^3+3(0.8)^4\\3(1.0)^4-4(0.8)(1.0)^3+(0.8)^4\end{bmatrix}$$

$$=\begin{bmatrix}0\\1\end{bmatrix}\frac{dT(x_6)}{dx}+\begin{bmatrix}-1\\0\end{bmatrix}\frac{dT(x_5)}{dx}+\frac{1000}{(12)(0.2)}\begin{bmatrix}1-2.0480+1.2288\\3-3.2000+0.4095\end{bmatrix}$$

$$=\begin{bmatrix}0\\1\end{bmatrix}\frac{dT(x_6)}{dx}+\begin{bmatrix}-1\\0\end{bmatrix}\frac{dT(x_5)}{dx}+\begin{bmatrix}75.3333\\87.2917\end{bmatrix}\begin{matrix}5\\6\end{matrix} \tag{1.61}$$

Next is to use the direct stiffness method to assemble both element stiffness matrices and element force vectors. This gives

$$\begin{bmatrix} 5 & -5 & 0 & 0 & 0 & 0 \\ -5 & 10 & -5 & 0 & 0 & 0 \\ 0 & -5 & 10 & -5 & 0 & 0 \\ 0 & 0 & -5 & 10 & -5 & 0 \\ 0 & 0 & 0 & -5 & 10 & -5 \\ 0 & 0 & 0 & 0 & -5 & 5 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} 0.667 - \dfrac{dT(x_1)}{dx} \\ 9.33 \\ 33.3 \\ 73.3 \\ 129.4 \\ 87.3 + \dfrac{dT(x_6)}{dx} \end{bmatrix} \tag{1.62}$$

We then modifyEq.(2.94) to impose the boundary conditions $T_1 = 0 \quad and \quad T_2 = 0$ which results in

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & -5 & 0 & 0 & 0 \\ 0 & -5 & 10 & -5 & 0 & 0 \\ 0 & 0 & -5 & 10 & -5 & 0 \\ 0 & 0 & 0 & -5 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 9.33 \\ 33.3 \\ 73.3 \\ 129.4 \\ 0.0 \end{bmatrix} \tag{1.63}$$

Solving Eq.(2.95) for the nodal degree of freedoms yields

$$\begin{aligned} T_1 &= 0 \\ T_2 &= 16.5 \\ T_3 &= 31.2 \\ T_4 &= 39.2 \\ T_5 &= 32.5 \\ T_6 &= 0 \end{aligned} \tag{1.64}$$

1.4 Results and conclusion

x	Exact solution	Point collocation	Sub-domain	Least squares	Rayleigh - Ritz	Finite element
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.2	16.5333	18.6480	21.3010	17.2815	17.1169	16.5000
0.4	31.2000	35.3239	40.8593	31.7126	31.5099	31.2000
0.6	39.2000	45.4872	53.5118	39.0396	38.9769	39.2000
0.8	32.5333	39.4949	47.2990	32.2197	32.3484	32.5000
1.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 1: comparison of the exact solution with the numerical solutions.

From thesummary of the results in the table 1,a survey of the values from five solution techniques shows that the finite element solution gives results closer to the exact solutions. Hence we conclude that the finite element method is an efficient numerical method for solving boundary value field problems.

1.5 References

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