AN EFFICIENT FINITE ELEMENT MODEL FOR TWO DIMENSIONAL FIELD PROBLEMS USING GALERKIN WEIGHTED RESIDUAL METHOD

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Abstract

An efficient numerical procedure of dealing with boundary value field problem is presented. The method is based on the finite element method. Its essence is the minimization of the error (residual) due to approximation in a weighted sense and is an element wise application of the Galerkin weighted method. The weighted residual integral gives a set of element algebraic equations, describing the variation of the function of interest at various discrete nodal points.

The assembly of the element equations using direct stiffness method gave a global system of equations (the model) which upon imposing the boundary conditions gave the desired nodal degree of freedom.

The solution and post process of finite element method of this study showed that once the stiffness matrix of a continuum is established and the boundary conditions specified, the continuum is solved uniquely.

The heat transfer problem was solved using our new model and the result obtained was seen to compare favorably with their closed form analytical solution.

Keywords: Finite element, Galerkin weighted residual, direct stiffness method

1. Introduction

A survey of occurrence in live shows that virtually every phenomenon in nature can be described with the aid of the law of physics in terms of algebraic, differential or integral equations relating various quantities of interest. This phenomenon is modeled through the mathematical formulation of the physical process as well as the numerical analysis of the mathematical model

Development of the mathematical model of a process is achieved through assumptions concerning how the process works. The derivation of the governing equations for most problems is not unduly difficult but the solution by exact method of analysis is a formidable task. This may be as a result of the region under consideration being so irregular that it is mathematically impossible to describe the boundary. The configuration may be composed of several different materials whose regions are mathematically difficult to describe. Problems involving anisotropic materials are usually difficult to solve analytically, as are equations having nonlinear terms. In such cases, approximate methods of analysis provide alternative means of finding solutions. Among these, the finite difference method [1],[2] and the variational methods such as the Rayleigh-Ritz and Galerkin methods are most frequently used in literature.

In the solution of a differential equation by a variational method, the equation is put into an equivalent weighted –integral form and the approximate solution $\dot{\vartheta}$ over the domain is assumed to be a linear combination $\left[\dot{\vartheta}(x) = \sum_j c_j \phi_j\right]$ of appropriately chosen approximation functions ϕ_j and undetermined coefficients, c_j. The coefficients c_j are determined such that the integral statement equivalent to the original differential equation is satisfied. Various variational methods, e.g, the Rayleigh-Ritz, point collocation, subdomain collocation, Least-squares and Galerkin methods , differ from each other in the choice of the integral form , weight functions, and /or approximation functions. They suffer from the disadvantage that the approximation functions for problems with arbitrary domains are difficult to construct.

Several work in literature shows that the Rayleigh Ritz and Galerkin methods gives results closer to the exact solution than the other variational methods , hence they are adopted in to the finite element method. invariably we can say that the finite element method is an element –wise application of the Rayleigh Ritz and Galerkin methods.

In this article, we shall be using the Galerkin finite element method to develop a model for two dimensional field problems.

2. The Mathematical problems

Significant class of physical problems representing phenomena such as heat transfer in solids, elastic torsion of prismatic bars, diffusion of pressure in porous media , flow of electric and magnetic potentials are known as field problems. They are related to that class of problems in which the governing partial differential equation is Poisson's equation. Their equilibrium conditions are all governed by Poisson's equation, in which the dependent variables are temperature, stress function, pressure, and potential or stream functions, respectively. The steady state field problems are governed by the quasi-harmonic equation:

$$D_x \frac{\partial^2 \emptyset}{\partial x^2} + D_y \frac{\partial^2 \emptyset}{\partial y^2} - G\emptyset + Q = 0 (1)$$

Subject to the boundary condition

$\emptyset = \overline{\emptyset}$; *value of* $\emptyset$ *prescribed on part of the boundary* Γ_1(2)

and

$$D_x \frac{\partial \phi}{\partial x} cos\Theta + D_y \frac{\partial \phi}{\partial y} sin\Theta = -\mathrm{M}\phi_b + \mathrm{S}$$

On the remaining part of the boundary, Γ_2 (*cauchy condition*)(3)

The mathematical problem is to formulate a model to handle the differential equation (1) subject to the boundary conditions eqn.(2) and eqn.(3).

3. The Finite element solution

We present the finite element solution of the problem according to the Galerkin weighted method. after discretizing the structure into triangular elements and assuming a linear variation of the field variable ϕ over each element using an interpolation polynomial of the form

$$\phi^{(e)} = \alpha_1 + \alpha_2 x + \alpha_3 y = N_i \Phi_i + N_j \Phi_j + N_k \Phi_k \qquad (4)$$

Where

$$N_i = \frac{1}{2A}[a_i + b_i x + c_i y]$$

$N_j = \frac{1}{2A}[a_j + b_j x + c_j y]$(5)

$$N_i = \frac{1}{2A}[a_k + b_k x + c_k y]$$

$$\begin{vmatrix} 1 & X_i & Y_i \\ 1 & X_j & Y_j \\ 1 & X_k & Y_k \end{vmatrix} = 2A \ (6)$$

and

$$a_i = X_j Y_k - X_k Y_j$$

$a_j = X_k Y_i - X_i Y_k$(7)

$$a_k = X_i Y_j - X_j Y_i$$

and

$$c_i = X_k - X_j$$

$c_j = X_i - X_k$(8)

$$c_k = X_j - X_i$$

$\Phi^{(e)} = \begin{Bmatrix} \Phi_i \\ \Phi_j \\ \Phi_k \end{Bmatrix}$=the nodal values(9)

Adopting a similar step applied for one dimensionalproblem [3], the element contribution to the systems is given by

$$\left\{R^{(e)}\right\} = -\int_A [N]^T \left(D_x \frac{\partial^2 \phi}{\partial x^2} + D_y \frac{\partial^2 \phi}{\partial y^2} - G\phi + Q \right) dA \quad (10)$$

Where $[N]$ is the row vector containing the element shape functions.

Applying the product rule for differentiation, the quantity

$$\frac{\partial}{\partial x}\left([N]^T \frac{\partial \phi}{\partial x} \right) \quad (11)$$

gives

$$\frac{\partial}{\partial x}\left([N]^T \frac{\partial \phi}{\partial x} \right) = [N]^T \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial [N]^T}{\partial x} \frac{\partial \phi}{\partial x} \quad (12)$$

Rearranging and substituting for $[N]^T \dfrac{\partial^2 \phi}{\partial x^2}$ in (11) produces

$$-\int_A [N]^T D_x \frac{\partial^2 \phi}{\partial x^2} dA = -\int_A D_x \frac{\partial}{\partial x}\left([N]^T \frac{\partial \phi}{\partial x} \right) dA + \int_A D_x \frac{\partial [N]^T}{\partial x} \frac{\partial \phi}{\partial x} \quad (13)$$

The first integral on the right-hand side of (13) can be replaced by an integral around the boundary using Green's theorem[4]. Application of the theorem yields

$$\int_A \frac{\partial}{\partial x}\left([N]^T \frac{\partial \phi}{\partial x}\right) dA = \int_\Gamma [N]^T \frac{\partial \phi}{\partial x} \cos\theta d\Gamma \qquad (14)$$

Where θ the angle to the outward normal and Γ is the element boundary. Substituting (14) into (13) gives the final relationship for the second-derivative term as

$$-\int_A D_x [N]^T \frac{\partial^2 \phi}{\partial x^2} dA = -\int_\Gamma D_x [N]^T \frac{\partial \phi}{\partial x} \cos\theta d\Gamma + \int_A D_x \frac{\partial [N]^T}{\partial x} \frac{\partial \phi}{\partial x} dA \qquad (15)$$

A similar set of operations starting with

$$\frac{\partial}{\partial y}\left([N]^T \frac{\partial \phi}{\partial y}\right)$$

Produces

$$-\int_A D_y [N]^T \frac{\partial^2 \phi}{\partial y^2} dA = -\int_\Gamma D_y [N]^T \frac{\partial \phi}{\partial y} \sin\theta d\Gamma + \int_A D_y \frac{\partial [N]^T}{\partial y} \frac{\partial \phi}{\partial y} dA \qquad (16)$$

Substitution of (15) and (16) into (11) gives

$$\begin{aligned}\left\{R^{(e)}\right\} = &-\int_\Gamma [N]^T \left(D_x \frac{\partial \phi}{\partial x}\cos\theta + D_y \frac{\partial \phi}{\partial y}\sin\theta\right) d\Gamma + \int_A \left(D_x \frac{\partial [N]^T}{\partial x}\frac{\partial \phi}{\partial x} + D_y \frac{\partial [N]^T}{dy}\frac{\partial \phi}{\partial y}\right) dA \\ &+\int_A G [N]^T \phi dA - \int_A Q [N]^T dA\end{aligned} \qquad (17)$$

Substituting for ϕ using the relationship

$$\phi^{(e)} = [N]\left\{\vec{\Phi}^{(e)}\right\} \qquad (18)$$

and rearranging gives

$$\left\{R^{(e)}\right\} = -\int_\Gamma [N]^T \left(D_x \frac{\partial \phi}{\partial x}\cos\theta + D_y \frac{\partial \phi}{\partial y}\sin\theta\right) d\Gamma + \left(\int_A \left(D_x \frac{\partial [N]^T}{\partial x}\frac{\partial [N]}{\partial x} + D_y \frac{[N]^T}{\partial y}\frac{\partial [N]}{\partial y}\right)\right) dA\left\{\vec{\Phi}^{(e)}\right\}$$

$$+\left(\int_A G[N]^T [N] dA\right)\{\vec{\Phi}^{(e)}\} - \int_A Q[N]^T \, dA \qquad (19)$$

Which has the general form

$$\{R^{(e)}\} = \{I^{(e)}\} + [k^{(e)}]\{\vec{\Phi}^{(e)}\} - \{f^{(e)}\} \qquad (20)$$

Where $\{I^{(e)}\} = -\int_\Gamma [N]^T \left(D_x \frac{\partial \phi}{\partial x}\cos\theta + D_y \frac{\partial \phi}{\partial y}\sin\theta\right) d\Gamma$ (21)

$$[k^{(e)}] = \int_A \left(D_x \frac{\partial [N]^T}{\partial x}\frac{\partial [N]}{\partial x} + D_y \frac{\partial [N]^T}{\partial y}\frac{\partial [N]}{\partial y}\right) dA + \int_A G[N]^T [N] dA \qquad (22)$$

and

$$\{f^{(e)}\} = \int_A Q[N]^T \, dA \qquad (23)$$

The first integral in (22) can be written more compactly by defining

$$[D] = \begin{bmatrix} D_x & 0 \\ 0 & D_y \end{bmatrix} \qquad (24)$$

and the gradient vector

$$\{gv\} = \begin{Bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \end{Bmatrix} = \begin{bmatrix} \frac{\partial [N]}{\partial x} \\ \frac{\partial [N]}{\partial y} \end{bmatrix} \{\Phi^{(e)}\} = [B]\{\Phi^{(e)}\} \qquad (25)$$

Hence

$$[B]^T = \begin{bmatrix} \frac{\partial [N]^T}{\partial x} & \frac{\partial [N]^T}{\partial y} \end{bmatrix} \qquad (26)$$

Using (24), (25) and (26), we have

$$\int_A [B]^T [D][B] dA = \int_A \left(D_x \frac{\partial [N]^T}{\partial x} \frac{\partial [N]}{\partial x} + D_y \frac{\partial [N]^T}{\partial y} \frac{\partial [N]}{\partial y} \right) dA \qquad (27)$$

The stiffness matrix for field problem is usually written as

$$\left[k^{(e)}\right] = \int_A [B]^T [D][B] dA + \int_A G[N]^T [N] dA \quad (28)$$

$$\left[k^{(e)}\right] = \left[k^{(e)}{}_D\right] + \left[k_G{}^{(e)}\right] \qquad (29)$$

4. Evaluation of the Element matrices

The scalar quantity ϕ is defined over a triangular region by

$$\phi^{(e)} = \begin{bmatrix} N_i & N_j & N_k \end{bmatrix} \left\{\Phi^{(e)}\right\} \quad (30)$$

Where N_i, N_j, N_k are given in (5)

The gradient vector for this element is

$$\{gv\} = \begin{bmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial x} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_j}{\partial y} & \frac{\partial N_k}{\partial y} \end{bmatrix} \left\{\Phi^{(e)}\right\}$$

$$= \frac{1}{2A} \begin{bmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} \left\{\Phi^{(e)}\right\} = [B]\left\{\Phi^{(e)}\right\} \qquad (31)$$

Therefore

$$\left[k_D{}^{(e)}\right] = \int_A [B]^T [D][B] dA = [B]^T [D][B] \int_A dA$$

$$= [B]^T [D][B] A \qquad (32)$$

Expanding the matrix products yields

$$\left[k_D{}^{(e)}\right] = \frac{D_x}{4A} \begin{bmatrix} b_i^2 & b_i b_j & b_i b_k \\ b_i b_j & b_j^2 & b_j b_k \\ b_i b_k & b_j b_k & b_k^2 \end{bmatrix} + \frac{D_y}{4A} \begin{bmatrix} c_i^2 & c_i c_j & c_i c_k \\ c_i c_j & c_j^2 & c_j c_k \\ c_i c_k & c_j c_k & c_k^2 \end{bmatrix} (33)$$

The second integral of (29) involves the shape functions. If we assume that G is constant within the element, this integral becomes

$$\left[k_G^{(e)}\right] = \int_A G[N]^T [N]\,dA = G\int_A \begin{Bmatrix} N_i \\ N_j \\ N_k \end{Bmatrix}\begin{bmatrix} N_i & N_j & N_k \end{bmatrix} dA$$

$$= G\int_A \begin{bmatrix} N_i^2 & N_iN_j & N_iN_k \\ N_iN_j & N_i^2 & N_jN_k \\ N_iN_k & N_jN_k & N^2{}_k \end{bmatrix} dA$$

$$= G\int_A \begin{bmatrix} L_1^2 & L_1L_2 & L_1L_3 \\ L_1L_2 & L_2^2 & L_2L_3 \\ L_1L_3 & L_2L_3 & L^2{}_3 \end{bmatrix} dA \qquad (34)$$

since $N_i = L_1$, $N_j = L_2$ and $N_k = L_3$ for the linear triangle. Using the natural coordinate transformation:

$$\int_A L^a{}_1 L^b{}_2 L^c{}_3\, dA = \frac{a!b!c!}{(a+b+c+2)!}2A \qquad \text{and} \qquad \int_\ell \ell_1{}^a \ell_2{}^b\, d\ell = \ell\frac{a!b!}{(a+b+1)!} \qquad (35)$$

Equation (34) becomes

$$\left[k_G^{(e)}\right] = \frac{GA}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad (36)$$

The element force vector is

$$\left\{f^{(e)}\right\} = \int_A Q[N]^T\, dA = Q\int_A \begin{Bmatrix} N_i \\ N_j \\ N_k \end{Bmatrix} dA = Q\int_A \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} dA = \frac{QA}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \quad (37)$$

5. Assemblage of element equation and imposition of boundary condition

Using the direct stiffness method, we can assemble the element equation to obtain a system of algebraic equation of the form

$$[K]\vec{\Phi} = F \qquad (38)$$

Where $[K]$, is the assembled stiffness matrix and F is the system force vectors.

To impose the boundary condition, recall from (21) the interelement vector $\mathrm{I}^{(e)}$

$$\left\{I^{(e)}\right\} = -\int_{\Gamma}[N]^T\left(D_x\frac{\partial\phi}{\partial x}\cos\Phi + D_y\frac{\partial\phi}{\partial y}\sin\Phi\right)d\Gamma \quad (39)$$

$$= \int_{\Gamma_{bc}}[N]^T\left(M\phi_b - S\right)d\Gamma \quad (40)$$

From the boundary condition (3)

Which can be separated into:

$$\left\{I_{bc}{}^{(e)}\right\} = \left(\int_{\Gamma_{bc}} M[N]^T[N]d\Gamma\right)\left\{\Phi^{(e)}\right\} - \int_{\Gamma_{bc}} S[N]^T\, d\Gamma \quad (41)$$

$$= \left[k_m{}^{(e)}\right]\left\{\Phi^{(e)}\right\} - \left\{f_s{}^{(e)}\right\} \quad (42)$$

Where

$$\left[k_m{}^{(e)}\right] = \int_{\Gamma_{bc}} M[N]^T[N]d\Gamma \quad (43)$$

and

$$\left\{f_s{}^{(e)}\right\} = \int_{\Gamma_{bc}} S[N]^T\, d\Gamma \quad (44)$$

6. EVALUATION OF THE ELEMENT INTEGRALS

$$\left[k_M{}^{(e)}\right] = \int_{\Gamma_{bc}} M[N]^T[N]d\Gamma = \int_{\Gamma_{bc}} M\begin{bmatrix}N_i\\N_j\\N_k\end{bmatrix}\begin{bmatrix}N_i & N_j & N_k\end{bmatrix}d\Gamma \quad (45)$$

$$\left[k_M{}^{(e)}\right] = \frac{ML_{ij}}{6}\begin{bmatrix}2 & 1 & 0\\1 & 2 & 0\\0 & 0 & 0\end{bmatrix} \quad (46)$$

Now we evaluate (43) for the triangular element for side ij

$$\left\{ f_S^{(e)} \right\} = \int_{\Gamma_{bc}} S[N]^T \, d\Gamma = L_{ij} \int_0^1 S \begin{Bmatrix} N_i \\ N_j \\ N_k \end{Bmatrix} d\ell_2$$

Since $N_k = 0$ along side ij

$$\left\{ f_S^{(e)} \right\} = L_{ij} \int_0^1 S \begin{Bmatrix} N_i \\ N_j \\ 0 \end{Bmatrix} d\ell_2 = L_{ij} \int_0^1 S \begin{Bmatrix} \ell_1 \\ \ell_2 \\ 0 \end{Bmatrix} d\ell_2 = \frac{SL_{ij}}{2} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \qquad (47)$$

Because the shape functions N_i and N_j reduces to $N_i = L_1 = \ell_1$ and $N_j = L_2 = \ell_2$ along side ij .

7. NUMERICAL EXAMPLE

For the two dimensional body shown in figure 1, we want to determine the n temperature distribution. The temperature at the left side of the body is maintained at $100^\circ F$ .The edges on the top and bottom of the body are insulated. There is heat convection from the right side with convection coefficient $h = 20 Btu/(h - ft^2 -^\circ F)$, the free stream temperature is $T_\infty = 50^\circ F$. The coefficients of thermal conductivity are $D_x = D_y = 25 Btu/(h - ft -^\circ F)$. The dimensions are shown in the figure 1. Assume a unit thickness.

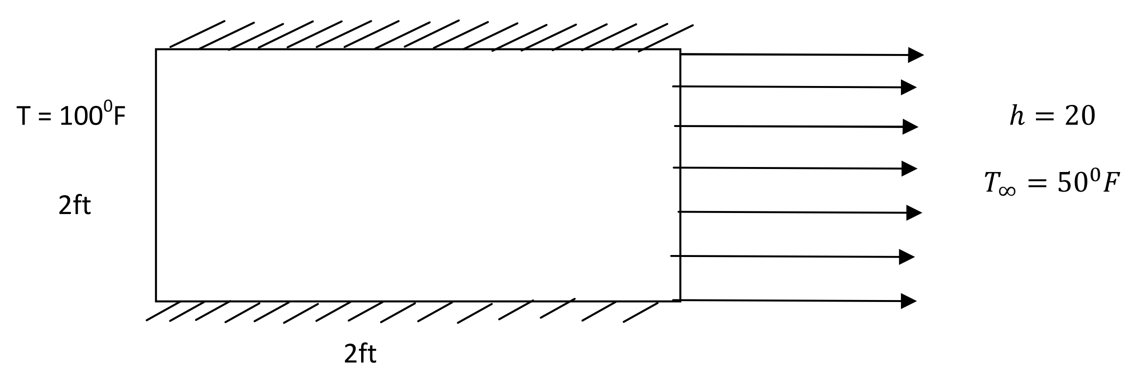

Figure 1: Two dimensional body subjected to temperature variation and convection

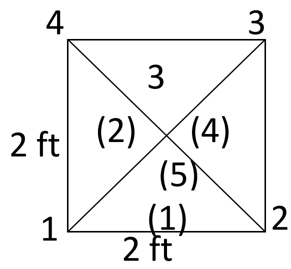

Figure 2: Discretized two – dimensional body of figure 1

Solution

The finite element discretization is shown in figure 2. We will use four triangular elements of equal size. There will be convective heat loss only over the right side of the body because the other faces are insulated. We now calculate the element stiffness matrices using (32) applied for all elements and using (47) applied for element 4 only, because convection is occurring only across one edge of element 4.

ELEMENT 1

The coordinates of the element 1 nodes are $x_1 = 0$, $y_1 = 0$, $x_2 = 2$,$y_2 = 0 x_5 = 1 and\ y_5 = 1$. hence

1 2 5

$$\left[k_C^{(1)}\right] = \begin{bmatrix} 12.5 & 0 & -12.5 \\ 0 & 12.5 & -12.5 \\ -12.5 & -12.5 & 25 \end{bmatrix} Btu/(h^{\circ}F) \qquad (48)$$

Where the numbers above the columns indicate the node numbers associated with the matrix

ELEMENT 2

The coordinates of the element 2 nodes are $x_1 = 0$, $y_1 = 0$, $x_5 = 1$, $y_5 = 1$, $x_4 = 0$ and $y_4 = 2$

1 5 4

$$\left[k_C^{(2)}\right] = \begin{bmatrix} 12.5 & -12.5 & 0 \\ -12.5 & 25 & -12.5 \\ 0 & -12.5 & 12.5 \end{bmatrix} Btu/(h^{\circ}F) \qquad (49)$$

ELEMENT 3

The coordinates of the element 3 nodes are

$x_4 = 0, y_4 = 2, x_5 = 1, y_5 = 1, x_3 = 2$ and $y_3 = 2$., hence

4 5 3

$$\left[k_C{}^{(3)}\right]=\begin{bmatrix}12.5 & -12.5 & 0\\ -12.5 & 25 & -12.5\\ 0 & -12.5 & 12.5\end{bmatrix} Btu/(h^{\circ}F) \qquad (50)$$

ELENENT 4

The coordinates of the element 4 nodes are $x_2 = 2, y_2 = 0, x_3 = 2, y_3 = 2, x_5 = 1$

and $y_5 = 1$, hence

2 3 5

$$\left[k_C{}^{(4)}\right]=\begin{bmatrix}12.5 & 0 & -12.5\\ 0 & 12.5 & -12.5\\ -12.5 & -12.5 & 25\end{bmatrix} Btu/(h^{\circ}F) \qquad (51)$$

For element 4, we have a convection contribution to the total stiffness matrix because side 2-3 is exposed to the free-stream temperature; Also $N_k = 0$, hence we obtain

$$\left[k_h{}^{(4)}\right]=\frac{hL_{ij}}{6}\begin{bmatrix}2 & 1 & 0\\ 1 & 2 & 0\\ 0 & 0 & 0\end{bmatrix} \qquad (52)$$

Where $L_{ij} = L_{2-3}$ is the side exposed to the free stream temperature. Hence

$$\left[k_h{}^{(4)}\right]=\frac{20\times 2}{6}\begin{bmatrix}2 & 1 & 0\\ 1 & 2 & 0\\ 0 & 0 & 0\end{bmatrix} \qquad (53)$$

Simplifying yields

2 3 5

$$\left[k_h{}^{(4)}\right]=\begin{bmatrix}13.3 & 6.67 & 0\\ 6.67 & 13.3 & 0\\ 0 & 0 & 0\end{bmatrix} \qquad (54)$$

Now $\left[k^{(4)}\right]=\left[k_C{}^{(4)}\right]+\left[k_h{}^{(4)}\right]$

2 3 5

$$= \begin{bmatrix} 25.83 & 6.67 & -12.5 \\ 6.67 & 25.83 & -12.5 \\ -12.5 & -12.5 & 25 \end{bmatrix} \qquad (55)$$

Superimposing the stiffness matrices , we obtain the total stiffness matrix for the body as

$$[K] = \begin{bmatrix} 25 & 0 & 0 & 0 & -25 \\ 0 & 38.33 & 6.67 & 0 & -25 \\ 0 & 6.67 & 38.33 & 0 & -25 \\ 0 & 0 & 0 & 25 & -25 \\ -25 & -25 & -25 & -25 & 100 \end{bmatrix} \qquad (56)$$

Next, we determine the element force matrices. Since the convective heat transfer occur only from side 2-3, element 4 is the only one that contributes nodal force; hence

$$\left\{ f^{(4)} \right\} = \begin{Bmatrix} f_2 \\ f_3 \\ f_5 \end{Bmatrix} = \frac{hT_\infty L_{2-3}}{2} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} \qquad (57)$$

Substituting the appropriate numerical values in equation (58) yields

$$= \frac{20 \times 50 \times 2}{2} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1000 \\ 1000 \\ 0 \end{Bmatrix} \frac{Btu}{h} \qquad (58)$$

The total assembled system of equations is

$$\begin{bmatrix} 25 & 0 & 0 & 0 & -25 \\ 0 & 38.33 & 6.67 & 0 & -25 \\ 0 & 6.67 & 38.33 & 0 & -25 \\ 0 & 0 & 0 & 25 & -25 \\ -25 & -25 & 25 & -25 & 100 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 1000 \\ 1000 \\ F_4 \\ 0 \end{Bmatrix} \qquad (59)$$

We have known nodal temperature boundary conditions of $t_1 = 100^\circ F$ and $t_4 = 100^\circ F$. We again modify the stiffness and force matrices as follows

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 38.33 & 6.67 & 0 & -25 \\ 0 & 6.67 & 38.33 & 0 & -25 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -25 & 25 & 0 & 100 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 1000 \\ 1000 \\ 100 \\ 5000 \end{Bmatrix} \qquad (60)$$

The resulting solution is given by

$$t_2 = 69.33^{\circ} F \ , \ t_3 = 69.33^{\circ} F \ , \ t_5 = 84.62^{\circ} F \qquad (61)$$

8. Concluding remarks

Discretizing the continuum into triangular elements and assuming a linear variation over the elements, the stiffness matrices and the force vectors were found, giving the Galerkin finite element model. The temperature distributions gotten in (62) agree with the physics of the problem.

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