# OPTIMAL EXPECTED VALUE OF ASSETS UNDER PARABOLIC EQUATION WITH MARKET PRICE OF RISK NOT ZERO

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**ABSTRACT**

This paper deals with optimal expected value of assets under parabolic equation when the market price of risk is not equal to zero. A seemingly Black-Scholes parabolic equation was obtained and then solved using Euler's substitution method when the market price of risk is not zero. We then used our result for the optimal prediction of the expected value of assets.

Keywords: *Fractal scaling exponent, Black-Scholes equation, Assets price return, optimal value, parabolic equation.*

MSC: 91G10, 91B28

## 1.0 INTRODUCTION

Options pricing is an important problem to which economists pay the exceptional attentions in the theoretical studies of international economics and finance engineering. One of the most appealing features of options (apart from the obvious chance of making extraordinary returns) is the possibility of easy speculation on the future behavior of a stock market price. The problems arising from the dynamic nature of stock prices has been addressed extensively by many authors[1, 2].Xiong[3] applied the wavelet to measure the fractal dimension of Chinese stock market. Muzy*et al*. [4] estimated the statistical self-similarity exponents from the data and made a quadratic fit for some low order moments. Several studies have examined the cyclic long-term dependence property of financial prices, including stock prices [5, 6]. These studies used the classical rescaled range (R/S) analysis, first proposedin [7] and later refined in [8, 9], among others. A significant long range dependence was found using R/S analysis for 200 daily stock returns of securities listed on the New York stock exchange, as studied in [5]. A problem with the classical R/S analysis is that the distribution of its regression-based test statistics is not well defined. As a result, Lo [10]proposed the use of a modified R/S procedure with improved robustness. The modified R/S procedure has been applied to study dynamic behavior of stock prices [10, 11]. Teverovsky *et al* and co-workers[12, 13]] have identified a number of problems associated with Lo's method. In particular, they showed that Lo's method has a strong preference for accepting the null hypothesis of no long range dependence. This happens even with long-range dependent synthetic data. To account for the long-range dependence observed in financial data, Cutland et al[14] proposed to replace Brownian motion with fractional Brownian motion as the building block of stochastic models for asset prices. An account of the historical development of these ideas can be found in[10, 8,15].A simple stochastic algorithm in a drifted financial derivative system for pricing an American option under Black-Scholes model was proposed in [16]

But in this paper, we consider the optimal expected value of assets under parabolic equation with market price of risk not zero. We use Euler's substitution method to solve our parabolic equation. We then use our result for the optimal prediction of the expected value of assets.

## 2.0 THE MODEL

Consider a portfolio comprising h unit of assets in long position and one unit of the option in short position. At time, T the value of the portfolio is

$$hP - V, \tag{1}$$

measured by the fractal index $C^{\phi}(E) - V^{\phi}(E) \neq 0$.

After an elapse of time, $\Delta t$, the value of the portfolio will change by the rate $h(\Delta P + D_1 \Delta t) - \Delta V$ in view of the dividend received on h units held. By Ito's lemma [17]this equals

$$h(uP\Delta t + \sigma P \Delta z + D_1 \Delta t) - \left\{ \left( \frac{\partial V}{\partial t} + \frac{\partial V}{\partial P} uP + \frac{1}{2} \frac{\partial^2 V}{\partial P^2} \sigma^2 P^2 \right) \Delta t + \frac{\partial V}{\partial P} \sigma P \Delta z \right\}$$

OR

$$\left\{(huP + hD_1) \ -\left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial P}uP + \frac{1}{2}\frac{\partial^2 V}{\partial P^2}\sigma^2 P^2\right)\right\}\Delta t + (h\sigma P - \frac{\partial V}{\partial P}\sigma P)\Delta z$$

If we take $h = \frac{\partial V}{\partial P}$ (2)

the uncertainty term disappears, thus the portfolio in this case is temporarily riskless. It should therefore grow in value by the riskless rate in force i.e.

$$\left\{(huP + hD_1) \ -\left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial P}uP + \frac{1}{2}\frac{\partial^2 V}{\partial P^2}\sigma^2 P^2\right)\right\}\Delta t = (hP - V)r\Delta t.$$

Thus

$$D_1\frac{\partial V}{\partial P} \ -\left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial P^2}\sigma^2 P^2\right) = (\frac{\partial V}{\partial P}P - V)r$$

So

$$\frac{\partial V}{\partial t} + (rP - D_1)\frac{\partial V}{\partial P} + \frac{1}{2}\frac{\partial^2 V}{\partial P^2}\sigma^2 P^2 = rV\ . \qquad (3)$$

**Proposition 1:** For $D_1 \neq 0$, the solution of equation (3) is given as:

$$V(P) = \left(\frac{\alpha q_n^2}{2P}\right)^{\beta}\left\{Ae^{\lambda_1\frac{\alpha q_n^2}{2P}} + Be^{\lambda_2\frac{\alpha q_n^2}{2P}}\right\}, \qquad (4)$$

where

$$\lambda_1 = -\frac{2}{z} \pm \sqrt{\frac{4}{z^2} + \frac{8r}{z^2\sigma^2}} \text{ and } \lambda_2 = \pm\frac{1}{z}\sqrt{4 + \frac{8r}{\sigma^2}}\ . \qquad (5)$$

**Proof**

We take

$$Z = \frac{\alpha}{P}; V(P) = Z^{\beta}W(Z) \qquad (6)$$

Thus

$$\frac{dz}{dP} = -\frac{\alpha}{P^2} = -\frac{1}{\alpha}Z^2$$

$$\frac{dV}{dP} = \frac{dV}{dZ}.\frac{dZ}{dP}$$

$$= -\frac{1}{\alpha} Z^2 (\beta Z^{\beta-1} W + Z^{\beta} \frac{dW}{dZ})$$

$$= -\frac{1}{\alpha} (\beta Z^{\beta+1} W + Z^{\beta+2} \frac{dW}{dZ}).$$

Hence

$$\frac{d^2V}{dP^2} = \frac{d}{dP} (\frac{dV}{dZ}).\frac{dZ}{dP}$$

$$= -\frac{1}{\alpha} Z^2 (\beta(\beta+1) Z^{\beta} W + \beta Z^{\beta+1} \frac{dW}{dZ} + (\beta+2) Z^{\beta+1} \frac{dW}{dZ} + Z^{\beta+2} \frac{d^2W}{dZ^2}).$$

In this case V is not dependent on $r$. Substituting into the given differential equation we have

$$rZ^{\beta} W = \frac{\sigma^2}{2} (\beta(\beta+1) Z^{\beta} W + \beta Z^{\beta+1} \frac{dW}{dZ} + (\beta+2) Z^{\beta+1} \frac{dW}{dZ} + Z^{\beta+2} \frac{d^2W}{dZ^2})$$
$$+ \left(\frac{r\alpha}{Z} - D_1\right) \left(\frac{-1}{\alpha}\right) (\beta Z^{\beta+1} W + Z^{\beta+2} \frac{dW}{dZ}). \qquad (7)$$

Cancelling by $Z^{\beta}$ and collecting like terms we have

$$0 = \frac{\sigma^2}{2} Z^2 \frac{d^2W}{dZ^2} + \frac{dW}{dZ} \left(\sigma^2 (\beta+1) Z - rZ + \frac{D_1}{\alpha} Z^2\right) + W \left(\frac{\sigma^2}{2} \beta(\beta+1) - r\beta + \beta \frac{D_1}{\alpha} Z\right) - rw$$

Or

$$0 = \frac{\sigma^2}{2} Z^2 \frac{d^2W}{dZ^2} + \frac{dW}{dZ} Z \left(\sigma^2 (\beta+1) - r + \frac{D_1}{\alpha} Z\right)$$

$$+ W \left(\frac{\sigma^2}{2} \beta(\beta+1) - r(\beta+1) + \beta \frac{D_1}{\alpha} Z\right).$$

Let

$$\beta = 0 \text{ and } = \frac{D_1}{\alpha} Z . \qquad (8)$$

We obtain

$$Z^2 \frac{d^2W}{dZ^2} + 2Z \frac{dW}{dZ} - \frac{2Wr}{\sigma^2} = 0 \qquad (9)$$

We then solve equation (9) by change of variable using Euler's substitution .

Let $Z = e^t$, then $\ln Z = t$, and $\frac{dt}{dZ} = \frac{1}{Z}$.

$$\frac{dW}{dZ} = \frac{dW}{dt}\frac{dt}{dZ} = \frac{1}{Z}\frac{dW}{dt}$$

and $$\frac{d^2W}{dZ^2} = \frac{d}{dZ}\left(\frac{1}{Z}\frac{dW}{dt}\right) = \frac{1}{Z}\frac{d}{dZ}\left(\frac{dW}{dt}\right) + \frac{dW}{dt}\frac{d}{dZ}\left(\frac{1}{Z}\right),$$

hence

$$\frac{d^2W}{dZ^2} = \frac{1}{Z^2}\left(\frac{d^2W}{dt^2} - \frac{dW}{dt}\right)$$

Substituting the above equations in equation (9) gives

$$\frac{d^2W}{dt^2} + \frac{dW}{dt} - \frac{2Wr}{\sigma^2} = 0 \qquad (10)$$

Equation (10) is linear with constant coefficient. Solving equation (10): let

$$W = e^{\lambda t}$$

be the solution of equation (10), hence $W' = \lambda e^{\lambda t}$ and $W'' = \lambda^2 e^{\lambda t}$.

Substituting in equation (10) gives $\lambda^2 e^{\lambda t} + \lambda e^{\lambda t} - \frac{2r}{\sigma^2} e^{\lambda t} = 0$.Our auxiliary equation becomes

$$\lambda^2 + \lambda - \frac{2r}{\sigma^2} = 0 \quad ,$$

and $$\lambda_1 = \frac{-1+\sqrt{1+\frac{8r}{\sigma^2}}}{2}, \qquad (11a)$$

$$\lambda_2 = \frac{-1-\sqrt{1+\frac{8r}{\sigma^2}}}{2}. \qquad (11b)$$

But $W = e^{\lambda t}$ and $t = lnZ$, hence, $W = Z^{\lambda}$.

Therefore,

$$W(Z) = AZ^{\lambda_1} + BZ^{\lambda_2}, \qquad (12)$$

where $A$ and $B$ are arbitrary constants and $\lambda_1$ and $\lambda_2$ are as defined in equations (11a) and (11b).

But

$$V(P) = Z^{\beta} W(Z) \text{ where } Z = \frac{\alpha}{P}$$

$$= \left(\frac{\alpha}{P}\right)^{\beta} \{AZ^{\lambda_1} + BZ^{\lambda_2}\}$$

$$= \left(\frac{\alpha}{P}\right)^{\beta} \left\{A\left(\frac{\alpha}{P}\right)^{\lambda_1} + B\left(\frac{\alpha}{P}\right)^{\lambda_2}\right\}.$$

Also, from [18],we have $\alpha = \frac{aq_n^2}{2}$, with $0 \le a \le \frac{4}{q_n^2}$, and $q_n > 0$ a Bessel function given as $\frac{J_n}{2-2(x)}$. Hence

$$V(P) = \left(\frac{aq_n^2}{2P}\right)^{\beta} \left\{A\left(\frac{aq_n^2}{2P}\right)^{\lambda_1} + B\left(\frac{aq_n^2}{2P}\right)^{\lambda_2}\right\} \qquad (13)$$

**CONCLUSION:** Considering equation (13), we observed that when $a = 0$, the equation becomes $V(P) = 0$, this signifies no signal. If $a = 4$, equation (13) becomes $V(P) = \left(\frac{2q_n^2}{P}\right)^{\beta} \left\{A\left(\frac{2q_n^2}{P}\right)^{\lambda_1} + B\left(\frac{2q_n^2}{P}\right)^{\lambda_2}\right\}$, this implies that there is signal. We now further look at it when $q = 1$ to have

$V(P) = \left(\frac{2}{P}\right)^{\beta} \left\{A\left(\frac{2}{P}\right)^{\lambda_1} + B\left(\frac{2}{P}\right)^{\lambda_2}\right\}$. Hence, if $\lambda_1\ and\ \lambda_2$ are negative, the investment output decays. On the other hand, if $\lambda_1 and\ \lambda_2$ are positive, the investment output grows.

## REFERENCES


[1] Black, F., & Karasinski, P. (1991). Bond and options pricing with short rate and lognormal. *Financial Analysis Journal*,47(4), 52-59.

[2] Black, F., & Scholes, M. (1973). The valuation of options and corporate liabilities. *Journal of Econometrics*,81,637-654.

[3] Xiong, Z., (2002). Estimating the fractal dimension of financial time Series by wavelets systems. *Engineering-Theory and Practice*,12, 48-53.

[4] Muzy, J., Delour, J., & Bacry, E., (2000). Modeling fluctuations of financial time series: from cascade process to stochastic volatility Model. Euro. Phys. *Journal B*, 17, 537-548.

[5] Greene, M. T., & Fielitz B. D. (1997). Long term dependence in common stock returns. *Journal of Financial Economics*,5, 339-349.

[6] Aydogan, K., & Booth, G. G. (1988). Are there long cycles in common stock returns? *Southern Economic Journal*,55,141-149.

[7] Hurst, H. E., (1951). Long term storage capacity of reservoir.*Transactions of the American Society of Civil Engineers*,116, 770-799.

[8] Mandelbrot, B. B., (1997). *Fractals and scaling in finance: Discontinuity, Concentration, Risk.* New York: Springer-Verlag.

[9] Mandelbrot, B. B., & Wallis, J. R. (1969). Robustness of the rescaled range in the measurement of non-cyclic long-run statistical dependence. *Water Resources Research*,5, 967-988.

[10] Lo, A. W., (1991). Long term memory in stock market prices.*Econometrica*,59, 1279-1313.

[11] Cheung, Y. W., Lai, K. S., & Lai, M. (1994). Are there long cycles in foreign stock returns? *Journal of International Financial Markets, Institutions and Money*, 3(1), 33-48.

[12] Teverovsky, V., Taqqu, M., & Willinger, W., (1999). A critical look at Lo's modified R/S statistic. *Journal of statistical planning and inference*, 80, 211-227.

[13] Willinger, W., Taqqu, M., & Teverovsky, V., (1999). Stock market prices and long-range dependence. *Finance and Stochastic*, 3, 1-13.

[14] Cutland, N., Kopp, P., & Willinger, W. (1995). Stock price returns and the Joseph effect: A fractal version of the Black-Scholes model. *Progress in Probability*, 36, 327-351.

[15] Shiryaev, A. N., (1999). *Essentials of stochastic finance*. Singapore: World Scientific.

[16] Osu, B. Oand Solomon, O.U (2015).A simple stochastic algorithm for the solution to PDE with financial application. J. of NAMP Volume 32: 125-132.

[17**]** Osu, B. Oand Olunkwa, C (2014). Existence of Optimal Parameters for a Non-linear Black-Scholes Option Pricing Model with Transaction Cost and Portfolio RiskMeasures. Journal of NAMP, 28(1): 469-474.

[18] Osu,O. B and Adindu-Dick, J .I (2014): Optimal prediction of the expected value of assets under fractal scaling exponent. *Applied Mathematics and Sciences: An International Journal (Math SJ)*,1(3): 41-51.