The use of Fractional Derivatives to Generalize Hooke's and Newton's Laws bythe Scott Blair's Model

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ABSTRACT
In this paper, some pioneering roles of the British scientist G.W. Scott Blair in the formation of the applications of fractional modelling in rheology are discussed. Further development of this branch of science and some other few aspects in which fractional calculus are applicable are also briefly investigated. We used two methods in the construction of fractional model of a viscoelastic fluid via Maxwell model. The first method is a direct one while in the second method the fractional elements are determined by three parameters which leads to the constitutive equation of Maxwell model.

KEYWORDS: Viscoelasticity, Rheology,Scott Blair's model.

INTRODUCTION

Interest has grown in the applications of fractional calculus in modelling of different phenomena in science, engineering, and many other fields of learning in the past 30 years. To see more prospects in the development of fractional models, it will be of great significance and value to understand how such models appear, and the work that was done by the pioneers. Some of the works that are the fundamental in fractional modelling are that of G.W. Scott Blair. His role and work on fractional modelling are of great importance and has been a base for researchers in the field of fractional calculus [1].

The field of viscoelasticity is one of the most broadly investigated and has the most extensive applications of fractional calculus because of the substantial design of new materials. Viscoelasticity material models are the first successful applications of fractional calculus. Some of the materials are mostly elastic while others have mainly viscous features. Though there are many materials that exhibit both elastic and viscous behaviour. The mechanics of these viscoelastic materials can be modelled as arrangements of lossless elastic springs and lossy viscous dampers (dash pots). For a given material the stress due to the elastic element is proportional to the strain and the stress due to the viscous element is proportional to the time rate of change of strain [2,3].

Recent work in fractional calculus viscoelastic modelling starts with the aim

Voigt model (a spring and a damper connected in parallel) and the Kelvin model (a spring and a damper in series) are the two most frequently used viscoelastic material models. These gives a practical explanation of relaxation, creep and stress rate dependence for some viscoelastic materials [2,3]. Though for many more complex materials, they are not sufficiently accurate. In this paper, we focus more on the contributions of Scott Blair to viscoelastic material and some other few aspects.

SOME BASICDEFINITIONSAND THEORETICAL ASPECTS OF FRACTIONAL CALCULUS

ORIGINSOFFRACTIONAL CALCULUS

In this section, in order to have good background of the topic, we present some basic definitions and theoretical aspects of fractional calculus which will be helpful in the cause of this work.

The origin of fractional calculus can be traced back to the seventeen century, almost the same time as when the foundation of classical calculus (differential and integral calculus) was laid by Newton and Leibnitz [4].

The symbol $\frac{d^n}{dx^n}f(x)$ denote as the nth derivative of the function f was introduced by Leibnitz. In one of his letters to de l'Hospital, he asked what mathematical theme will

$\frac{d^n}{dx^n}f(x)$ have if n is not an integer?

L'Hospital replied him: What is $\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}f$ equal to when $f(x)=x$? It has been accepted universally that this letter from de l'Hospital to Leibnitz in 1695 gave birth to fractional calculus [4,5].

SOME BASIC DEFINITIONS OF FRACTIONAL CALCULUS

Here, we present some basic definitions of fractional calculus which are very useful in this work.

(i) The Riemann-Liouville definition: The definition of fractional α order integral of $f(t)$ function defined by Riemann-Liouville is given by

$$ {}_{t_0}D_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau, \qquad (1) $$

where t_0 is the lower integration limit, α is between zero and one $0 \le \alpha \le 1$, and Γ(α) represents the gamma function of the α argument, which can be calculated as

$$ \Gamma(\alpha) = \int_0^{\infty} e^{-\tau} \tau^{\alpha-1} f(\tau) d\tau, \ (2) $$

Representing the m order derivative as $D^m, m \in \mathrm{R}^+$, the fractional α order derivative Riemann-Lioville's definition yields

$$ {}_{t_0}D_t^{\alpha} f(t) = d^m {}_{t_0}I^{m-\alpha} f(t), (3) $$

which can also be written as

$$ {}_{t_0}D_t^{\alpha} f(t) = \frac{d^m}{dt^m} \left[\frac{1}{\Gamma(m-t_0)} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau \right], m-1 < \alpha < m, (4) $$

which satisfies the Laplace transform

$$ {}_{t_0}D_t^{\alpha} f(t) = \frac{1}{s^{m-\alpha}} \ell\{ d^m f(t) \} $$

$$ {}_{t_0}D_t^{\alpha} f(t) = s^{\alpha-m} \left(s^m \ell\{ f(t) \} - \sum_{k=0}^{m-1} s^{m-k} f^{k-1} \right) $$

$$ \ell\{ {}_{t_0}D_t^{\alpha} f(t) \} = s^{\alpha} F(s) - \sum_{k=0}^{m-1} s^k f^{\alpha-k-1}(t_0), \qquad (5) $$

where $f^{(\cdot)}(t_0)$ is the value of the $(\cdot)$ order of derivative at the integration lower limit t_0[6].

(ii) The Caputo's definition: Let a function $f(x)$ be defined on a semi-open interval $(a,b]$ and let $0 \le m-1 < \alpha \le m, m = 1,2,...,$ then the fractional derivative of order $\alpha > 0$ of the function is $f(t)$ defined by

\begin{equation}

$$ D_*^{q} f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^m(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau \qquad (6) $$

In Caputo's idea, the above definition means that a function $f(t)$ accept the fractional derivative of order if it $\alpha > 0$ has the ordinary derivative of order [α]+1 and if α is not integer. This shows that Caputo's idea of fractional derivative is defined for a smaller class of functions than the fractional derivatives in Riemann and Grunvald-Letnikov ideas. It is notable that when using the Caputo's fractional derivative for constitutive relations of viscoelasticity, the initial conditions are written in terms of ordinary derivatives of integral order of initial functions explicitly [5].

(iii) The Grunvald-Letnikov definition: Let a function be de $f(x)$ fined on a semi-open interval $(a,b]$, where a is any finite real number; then the differential-integral of order q of the function $f(x)$ at any point $x \le (a,b]$ is defined in [5] as

$$ {}_aD_x^{\,q} f(x) = \lim_{N\to\infty}\left\{ \frac{\left[\frac{x-a^{-q}}{N}\right]}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f\left(x - j\left[\frac{x-a}{N}\right]\right) \right\} \quad (7) $$

Where q is any infinite real number. For q is taken to be a nonnegative integer, and Γ(-q) becomes infinite. The formula

$$ \frac{\Gamma(-n)}{\Gamma(-N)} = (-1)^{N-n}\frac{N!}{n!}, $$

Holds for any nonnegative integers N and n, implies that the ratio $\frac{\Gamma(j-q)}{\Gamma(-q)}$, is always finite.

${}_aD_q^{\,x}$ for q ≤ 0 is an operation generally called the operation of fractional integration and is denoted as J^{-q}; for q, it is called operation of fractional differentiation when it is denoted as D^q [5].

RHEOLOGY AND VISCOELASTICITY

Rheology

Rheology is the study of the flow of matter, primarily in a liquid state but also solids' or solids under conditions in which they respond with plastic flow rather than deforming elastically in response to an applied force. The name ``Rheology'' was coined by Eugune Bingham, for this branch of science and he gave the definition of it : ``The science of deformation and flow of matter'' [7], motivated by Heraclitus' quote ``everything flows''.

The ancient Egyptian scientist Amenemhet (ac 1600 BC), made the earliest application of viscosity effect, so can be considered as the first rheologist [8].

The recognizable in rheology are deformations or strains, that change with time. The changes of strains in time form a flow. Thus, these changes are generally connected with inner flow of certain kind. State of stress are deduced either from the relative strain behaviour of complex or simple systems in interaction or form the reaction of a known mass in gravitational field. In application, rheology is concern with extending of the continuance of mechanics to distinguish flow of material, which shows a combination of elastic, plastics and viscous characteristics by properly combining elasticity and Newtonian fluid mechanics. These days rheology basically account for the behaviour of non-Newtonian fluids, by specifying the least number of functions that are needed to relate stress with rate of change of strain rates. This type of fluids is called Newtonian because Newton introduced the concept of viscosity [1].

Viscoelasticity

Viscoelasticity is when materials exhibit both viscous and elastic characteristics when they undergo deformation. It is important to point out that viscoelasticity is not plasticity, which it is often confused with. A viscoelastic material do return to its original shape after any deforming force is removed (showing an elastic response) though it may take time to do so (that is, it response to a viscous component). In material or visible measuring, stresses (σ), strains (ε) or their differentials with respect to time $\sigma(t), \varepsilon(t)$ are usually held constant, leaving either a length to be measured, or the time (t) [8].

There is a group of fluids known as Newtonian fluid, that is characterized by a coefficient of viscosity for a specific temperature. These fluids were detected by Newton who suggested the definition of resistance (or viscosity in recent language) of an ideal fluid.

Viscoelasticity can be expressed mathematically as shown below :
Elastic
(i)Stress: Force per unit area of particles exerted.
(ii)Strain: Measurement of deformation.
(iii)Strings (Hooke's law: Figure 1): $\sigma = E\varepsilon$.
Where σ : stress, ε : strain and E : Elastic modulus.

Viscous
(i)Stress: Force per unit area of particles exerted.
(ii)Strain rate: The rate at which strain occurs. It is the time rate of change of strain.
(iii)Dashpots (Newtonian fluid: Figure 2): $\sigma(t) = \eta \frac{d\varepsilon}{dt}$.
Where $\sigma(t)$: stress, $\varepsilon(t)$: strain rate and η :viscosity [9].

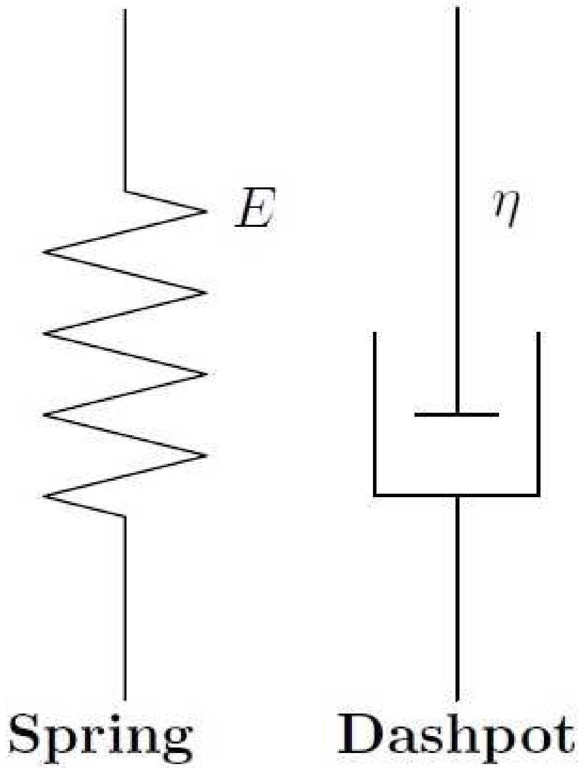

Figure 1: Hooke's element(Springs).   Figure 2: Newton's element (Dashpot).

SCOTT BLAIR FRACTIONAL CALCULUS ELEMENT

Scott Blair's fractional model

Newton's law describes a proportionality of the stress to the 1-derivative of strain, Hooke's law can be appraise as a proportionality of the stress to the 0-derivative of the strain. Thus, it is rational to introduce a more general model, a proportionality of the stress to the α-derivative of the strain where α ∈ (0,1) [10].

Scott Blair was the scientist that proposed the constitutive equation for a viscoelastic material whose mechanical properties are in between the a pure elastic solid (Hooke model) and a pure viscous liquid (Newton model). The Scott Blair's model is the fractional generalization of Hooke's model and Newton's model that is related in equation (8), and introducing the constant $\tau = \frac{\eta}{E}$ that warrant a good chance with the ideal solid for α = 0 and with the ideal fluid for α = 1, we have [10]

$$\sigma(t) = E\tau^{\alpha} D_0^{\alpha} \in (t). \qquad (8)$$

Graphs for various values of α are plotted for the relaxation modulus which is modulus obtained from the stress-relaxation observed during the decrease in stress in response to the same amount of strain generated in the structure and the stress creep compliances which is the tendency of a solid material to move slowly or deform permanently under the influence of mechanical stress (Figure 3 and Figure 4). Their calculations can be perform by Laplace transform:

$$G_B(t) = \frac{E\tau^{\alpha}}{\Gamma(1-\alpha)} t^{-\alpha}$$

$$J_B(t) = \frac{1}{E\tau^{\alpha}\Gamma(1-\alpha)} t^{\alpha}$$

Where $G_B(t)$ is the Blair's relaxation modulus function and $J_B(t)$ is the Blair's stress creep compliance function.

The fractional model gives a qualitative practical response functions. Here the infinite value of $G_B(t)$ for $t = 0$, there is slow grow of the strain for the step stress and the continuity of $G_B(t)$ and there is the stressrelaxation. Since close power-law stress relaxation was observed in real materials, this encourages more development of fractional model [10].

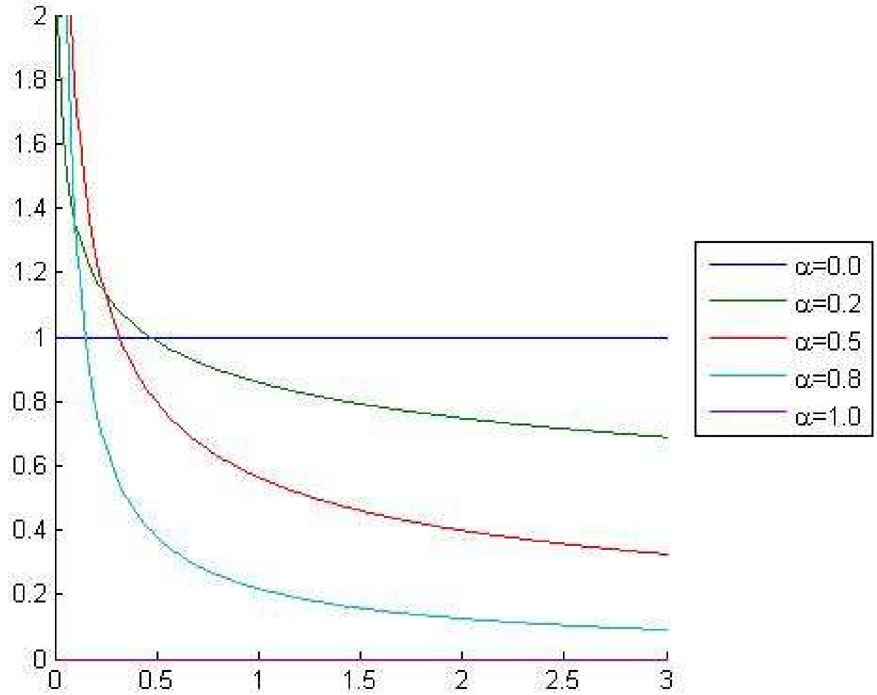

Figure 3: The relaxation modulus for Blair's model for various α $(E = 1, \tau = 1)$.

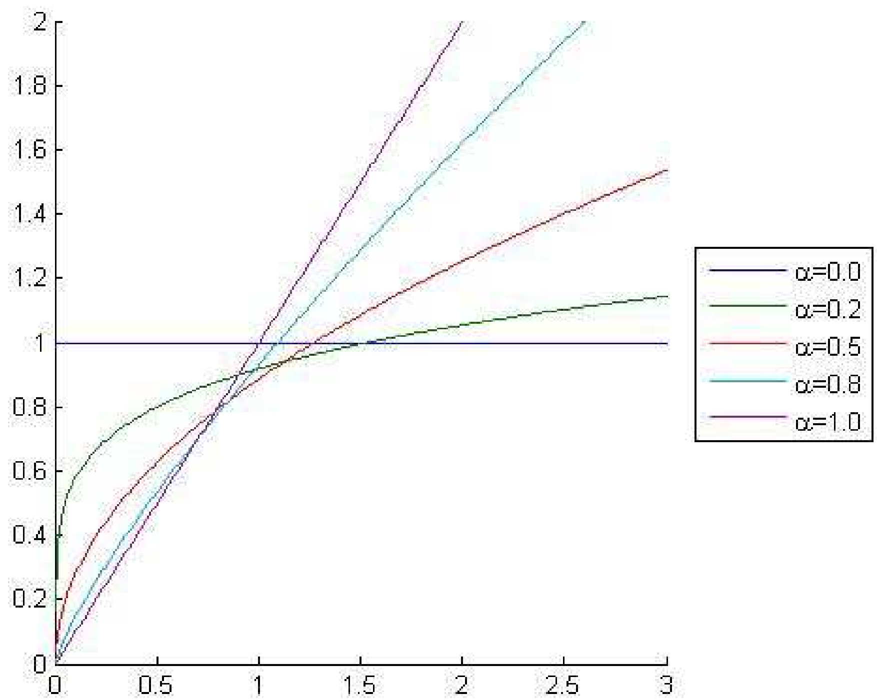

Figure 4: The stress creep compliance for Blair's model for various α $(E = 1, \tau = 1)$.

CONSTRUCTION OF A FRACTIONAL MODEL FOR VISCOELASTIC FLUID

It is a common tradition in rheology to manipulate representations instead of suited equations. Hooke elastic element is represented as a spring in Figure 1, while the Newton viscous element is represented as a dashpot in Figure 2. The Hooke elastic element and the Newton viscous element were combined at first with the intention of combining both properties. The possibilities of combining them are the parallel and the serial connection but for this construction, we are interested in that of the serial connection of the two basic elements (spring and dashpot), which gives the Maxwell's model of viscoelasticity.
For the Maxwell's model, we describe the relationship as [11]

$$\frac{d\varepsilon}{dt} = \frac{1}{E}\frac{d\sigma}{dt} + \frac{\sigma}{\mu} \qquad (9)$$

Where σ = constant $\Rightarrow \frac{d\varepsilon}{dt}$ = constant.

In the construction of fractional model, two methods are commonly use. The direct method is when the time derivatives of an integer order is replace by the Riemann-Liouville fractional calculus operators [12]. For example, we can construct the fractional Maxwell model through replacement that is by rearranging, substituting $\lambda = \frac{\mu}{E}$ and adding the fractional parameters α and β into (9)

$$\frac{d\varepsilon}{dt} = \frac{1}{E}\frac{d\sigma}{dt} + \frac{\sigma}{\mu}$$

$$\mu\frac{d\varepsilon}{dt} = \frac{\mu}{E}\sigma\ dt + \sigma$$

Now substituting $\lambda = \frac{\mu}{E}$ and adding the fractional parameters, we have

$$\sigma + \lambda^{\alpha}\frac{d^{\alpha}\sigma}{dt^{\alpha}} = E\lambda^{\beta}\frac{d^{\beta}\varepsilon}{dt^{\beta}} \qquad (10)$$

here α is the shear stress, ε is the shear strain, E is a shear modulus, $\lambda = \frac{\mu}{E}$ is the relaxation time, μ is the constant viscosity coefficient, and α and β are the fractional parameters which satisfies the range $0 < \alpha < 1,\ 0 < \beta < 1$. This method is easy, though its physical meaning is open to more than one interpretation, and to satisfy physical meaning (10) must have the restriction $\alpha \leq \beta$ [12].
In the second method there is assurance that the fractional model obtain through it will have correct physical meaning.

We begin by introducing the stress-strain relation with fractional order derivative

$$\sigma(t) = E\lambda^{\tau} \frac{d^{\tau}\varepsilon}{dt^{\tau}}, (0 < \tau < 1) \tag{11}$$

$$G(t) = \frac{E}{\Gamma(1-\tau)} \left(\frac{t}{\tau} \right)^{-\tau}, \tag{12}$$

where $G(t)$ is the relaxation modulus and $\Gamma(\cdot)$ is the Euler gamma function. The model is also called Scott Blair's model, and this can be shown by orderly arrangement of the springs and dashpots in form of trees. The model can also be presented as an interpolation between Hooke's law $(\lambda = 0)$ and Newton's law $(\lambda = 1)$ [7, 12].

To construct the fractional model, the fractional element defined as the mechanical elements that obey(11) is introduced. A fractional element is determined by three parameters (τ, E, λ) and it symbolized a triangle as shown in Figure 5(c). To construct the viscoelastic models, the newly introduced element Figure 5(c) is assumed to have the same status as the spring and dashpot in Figures 5(b) and 5(c).

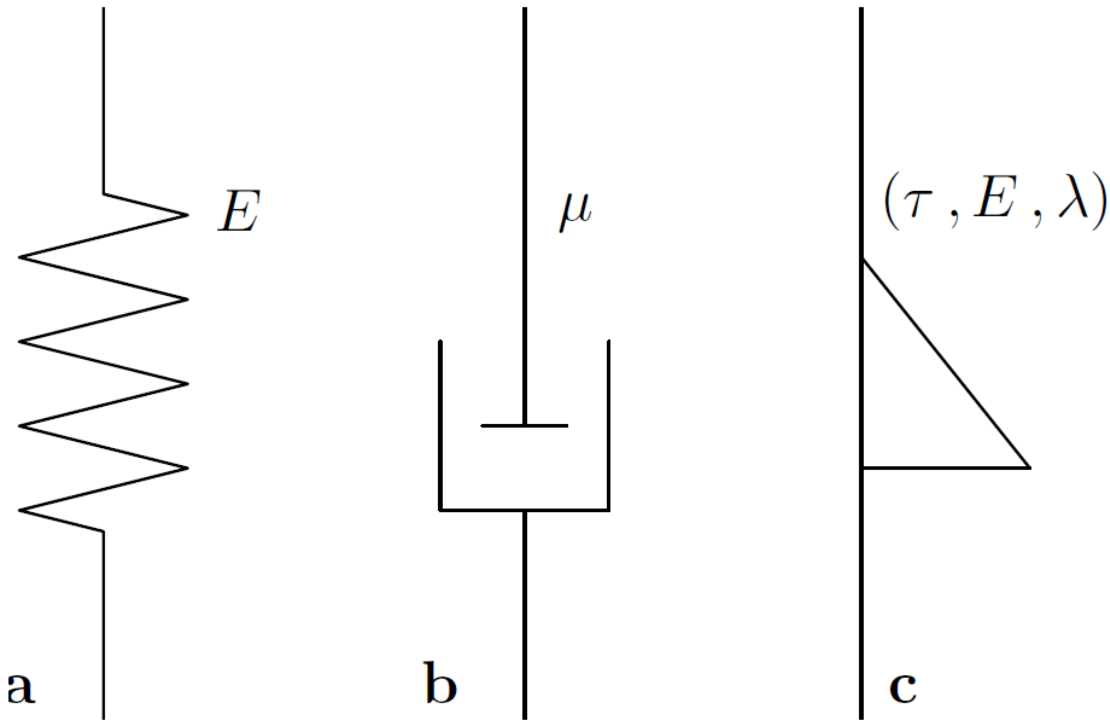

Figure 5: (a) Elastic element; (b) Viscous element and (c) Fractional element.

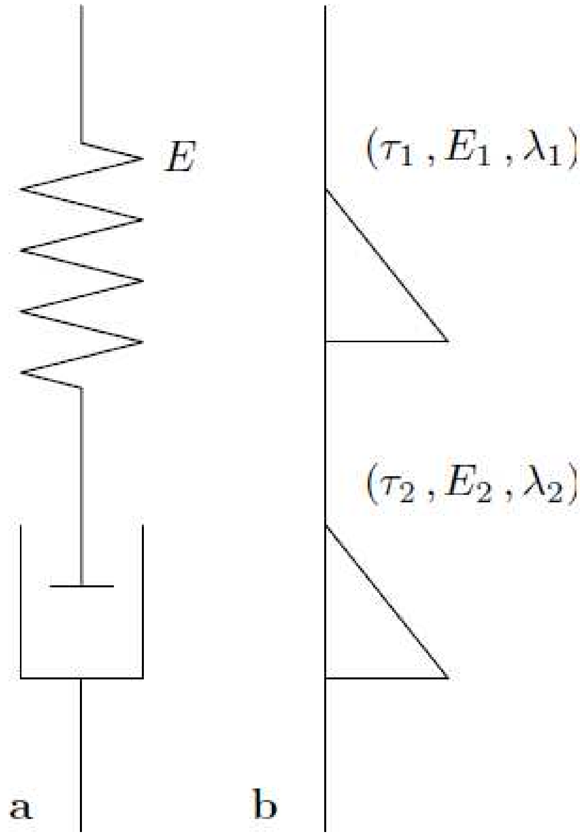

Figure 6: (a) The classical Maxwell model and (b) The fractional Maxwell model.

The ordinary Maxwell model was generalized as shown in Figure 6(a). Fractional elements with the parameters (τ_1, E_1, λ_1) and (τ_2, E_2, λ_2) and were used to replaced the two classical elements in figure 6(b) [12].

The fractional elements obey the stress-strain relations: $\sigma_1(t) = E_1 \lambda_1^{\tau_1} d^{\tau_1} \dfrac{\varepsilon_1(t)}{dt^{\tau_1}}$ and $\sigma_2(t) = E_2 \lambda_2^{\tau_2} d^{\tau_2} \dfrac{\varepsilon_2(t)}{dt^{\tau_2}}$.Fitting to series connection of the elements, we obtain

$$\sigma(t) + \frac{E_1 \lambda_1^{\tau_1}}{E_2 \lambda_2^{\tau_2}} \frac{d^{\tau_1-\tau_2}\sigma(t)}{dt^{\tau_1-\tau_2}} = E_1 \tau_1^{\tau_1} \frac{d^{\tau_1}\varepsilon(t)}{dt^{\tau_1}}. \qquad (13)$$

We assume without loss of generality that $\tau_1 > \tau_2$. Introducing the new variables

$\lambda = \left(\dfrac{E_1 \lambda_1^{\tau_1}}{E_2 \lambda_2^{\tau_2}} \right)^{\frac{1}{\tau_1-\tau_2}}$, and $E = E_1 \left(\dfrac{\lambda_1}{\lambda} \right)^{\tau_1}$ in (13), gives

$$\sigma(t) + \lambda^{\tau_1-\tau_2} \frac{d^{\tau_1-\tau_2}\sigma(t)}{dt^{\tau_1-\tau_2}} = E \tau^{\tau_1} \frac{d^{\tau_1}\varepsilon(t)}{dt^{\tau_1}} \qquad (14)$$

which is the constitutive equation of the fractional Maxwell model. Substituting $\tau_1 - \tau_2$ for α and τ_1 for β, we get the same equation as (10)

$$\sigma + \lambda^{\alpha} \frac{d^{\alpha}\sigma}{dt^{\alpha}} = E\lambda^{\beta} \frac{d^{\beta}\varepsilon}{dt^{\beta}} \qquad (15)$$

Since $\tau_1 - \tau_2$ is always less than τ_1 for $\tau_1 \geq 0$ and $\tau_2 \geq 0$, then the condition $\alpha \leq \beta$ must be used in (15) [12].

FRACTIONAL CONSTITUTIVE LAW

In this section we introduced some fundamental fractional inherent materials. The constitutive law relating stress $\sigma(t)$ and strain $\gamma(t)$ may be derived, and also to show that the Caputo fractional derivative and the Riemann-Liouville fractional integral can be generated in the process. Starting from the knowledge of the relaxation function $G(t)$ which represent the stress history for the assigned stress history $\gamma(t) = U(t)$, being $U(t)$ the unit step function. Boltzmann superposition principle gives

$$\sigma(t) = \int_0^t G(t-\tau)d\gamma(\tau) = \int_0^t G(t-\tau)\dot{\gamma}(\tau)d\tau \ . \qquad (16)$$

Equation (16) is valid for $\gamma(0) = 0$ [13]. The inverse of the constitutive law may faced starting from the creep function $J(t)$ which is the strain history for the assigned load history $\sigma(t) = U(t)$. The Boltzmann superposition principle for $\sigma(0) = 0$ and is written as

$$\gamma(t) = \int_0^t G(t-\tau)d\sigma(\tau) = \int_0^t G(t-\tau)\dot{\sigma}(\tau)d\tau \qquad (17)$$

It is obvious that from (16) and (17), $G(t)$ and $J(t)$ are related by the following relationship in the Laplace domain: Re-writing (16) and (17) using the convolution form and with the technique of the Laplace transform according to the notation

$$f(t) * g(t) \div \tilde{f}(\mathrm{s})\tilde{g}(\mathrm{s})$$

weshow that (16) and (17) yields

$$\tilde{\sigma}(\mathrm{t}) = \mathrm{s}\tilde{G}(\mathrm{s})\tilde{\gamma}(\mathrm{s})$$

$$\tilde{\gamma}(\mathrm{s}) = \mathrm{s}\tilde{J}(\mathrm{s})\tilde{\sigma}(\mathrm{s})$$

The basic relation between J $G(t)$ and $J(t)$ from the reciprocity relation in the Laplace domain

$$s\tilde{G}(s) = \frac{1}{s\tilde{J}(s)} \Leftrightarrow \tilde{G}(s)\tilde{J}(s)\frac{1}{s^2} \quad (18)$$

Where $\tilde{G}(s) = \ell\{G(s);s\}, \tilde{J}(s) = \ell\{J(s);s\}$, and s is the Laplace complex parameter.
Experimental test is the best fitting process in which relaxation function and $G(t)$ creep function $J(t)$ are obtained [13].

In [14] Nutting observed that every real material give a relaxation function well fitted by a power-law of the type

$$G(t) = \frac{C(\beta)}{\Gamma(1-\beta)} t^{-\beta}; 0 \le \beta \le 1, \qquad (19)$$

where $C(\beta)$ and β are the parameters depending on the material at hand and $\Gamma(\cdot)$ is the Euler gamma function.

Just as we assume for $G(t)$ the power-law decay expressed in (19), so also by assuming (17) the creep function $J(t)$ is obtained in the form

$$J(t) = \frac{t^{\beta}}{C(\beta)\Gamma(1+\beta)}. \qquad (20)$$

By substituting (19) in (16) and (20) substituting in (17) we have

$$\sigma(t) = \frac{C(\beta)}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} \dot{\gamma}(\tau) d\tau$$

$$\sigma(t) = C(\beta)\left({}_c D_{0+}^{\beta} \gamma\right)(t), \qquad (21)$$

$$\gamma(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \sigma(\tau) d\tau$$

$$\gamma(t) = \frac{1}{C(\beta)} \left(I_{0+}^{\beta} \sigma\right)(t), \qquad (22)$$

where $\left({}_c D_{0+}^{\beta} \gamma\right)(t)$ is the Caputo fractional derivative defined as

$$\left({}_c D_{0+}^{\beta} \gamma\right)(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} \dot{\gamma}(\tau) d\tau, \qquad (23)$$

and $\left(I_{0+}^{\beta} \gamma\right)(t)$ is the Riemann-Liouville fractional integral defined as

$$\left(I_{0+}^{\beta} \gamma\right)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \sigma(\tau) d\tau. \qquad (24)$$

The Caputo's definition in (23) is equivalent to the Riemann-Liouville fractional derivative if the initial condition is well recorded.

CONCLUSION

We were able to show that the Scott Blair's model is the fractional generalization of Hooke's model and Newton's model. Construction of a fractional model for viscoelastic fluid is also investigated by the direct method where the time derivatives of an integer order is replaced by the Riemann-Liouville fractional calculus operators, and fractional elements with a family of parameters are used to replaced some classical elements in the second method.

The Caputo fractional derivative and the Riemann-Liouville fractional integral were generated in the process of the derivation of the constitutive law relating stress $\sigma(t)$ and strain $\gamma(t)$.

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