

Some Properties of a Class of Salagean-Type P -valent Functions

¹ Oyekan Ezekiel Abiodun, ² Awoyale Olusegun and ³ Kehinde Rotimi

¹ Department of Mathematical Sciences, Ondo State University of Science and Technology, P.M.B. 353, Okitipupa, Ondo State-Nigeria.

² Department of Mathematics, Federal College of Education, Kontagora, Niger State-Nigeria.

³ Department of Mathematics, Federal University, Lokoja-Nigeria.

Corresponding Author: Oyekan, Ezekiel Abiodun, E-mail: ea.oyekan@osustech.edu.ng, Tel.: +234 8034772630

Abstract

In (2008) Güneş introduced and investigated the subclass $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$ consisting of analytic and p -valent functions with negative coefficients using a differential operator $D^\lambda = p^\lambda z^p - \sum_{k=1}^{\infty} (p+k)^\lambda a_{p+k} z^{p+k}$ and derived the coefficient inequalities, distortion theorem and extreme points for $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$. The purpose of present work is to derive some interesting properties other than those obtained by Güneş for the subclass. In particular, we derive quasi-Hadamard product (quasi-convolution) property, inclusion theorem, radius of close-to-convexity, star-likeness and convexity properties for $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$.

2010 Mathematics Subject Classification. 30C45, 26A33

Keywords and Phrases: Analytic, p -valent, Quasi-Hadamard product, Salagean operator, close-to-convex function, coefficient inequalities.

1.0 INTRODUCTION

A function $f(z)$ meromorphic in a domain D is said to be p -valent in D (or multivalent of order p in D) if for each w_0 (infinity included) the equation $f(z) = w_0$ has at most p roots in D (where the roots are counted in accordance with their multiplicity) and if there is some w_1 such that the equation $f(z) = w_1$ has exactly p roots in D [1].

The class of p -valent functions has been widely studied. In fact, Aouf [2] and Hossen [3] studied the subclasses of p -valent functions of order α and type β . In recent years, several interesting classes (and or subclasses) of p -valent analytic functions have been introduced and investigated [4 5 6 7 8 9 10 11 12].

Let $\mathbf{A}(p)$ denote the class of functions defined by

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}, \quad p \in \mathfrak{N} \quad (1.1)$$

which are analytic and p -valent in the open unit disk $U = \{z \in \mathbf{C}; |z| < 1\}$.

Let T_p^* denote the subclass of $\mathbf{A}(p)$ whose members have the form given by

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad , \quad a_{p+k} \geq 0, p \in \mathfrak{R}. \quad (1.2)$$

It is well known that the Hadamard Product (or Convolution) of two analytic functions, $f(z)$ defined by (1.1) and $g(z)$ given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \quad , \quad b_{p+k} \geq 0, p \in \mathfrak{R}. \quad (1.3)$$

is denoted by $(p * g)(z)$ and defined by

$$(p * g)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k} \quad , \quad p \in \mathfrak{R}. \quad (1.4)$$

Similarly, the modified Hadamard product of the function $f(z)$ defined by (1.2) and $g(z) \in T_p^*$ given by

$$g(z) = z^p - \sum_{k=1}^{\infty} b_{p+k} z^{p+k} \quad , \quad b_{p+k} \geq 0, p \in \mathfrak{R}. \quad (1.5)$$

is denoted by denoted by $(p * g)(z)$ and defined by

$$(p * g)(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k} \quad , \quad p \in \mathfrak{R}. \quad (1.6)$$

This definition follows from the similar definition by Mugur Acu [13].

In [12], Güneş introduced the differential operator $D^\lambda f(z)$: $D^\lambda f(z) = D(D^{\lambda-1} f(z)) = p^\lambda z^p - \sum_{k=1}^{\infty} (p+k)^\lambda a_{p+k} z^{p+k}$.

By making use of this differential operator, he defined the class $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$:

$$T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda) := \left\{ f \in T_p^* : \left| \frac{z(D^\lambda f(z))' - pD^\lambda g(z)}{(A-B)(p-\beta)D^\lambda g(z) - B[z(D^\lambda f(z))' - pD^\lambda g(z)]} \right| < \gamma, \quad z \in U, \right\} \quad (1.7)$$

for $g(z) \in T_p^*$, $-1 \leq B < A \leq 1$, $-1 \leq B < 0$, $0 \leq \alpha < p$, $0 \leq \beta < p$, and $0 < \gamma < 1$.

For the class $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$, properties such as coefficient inequalities, distortion theorem and extreme points were discussed. In the present work, we derive other properties of the class $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$.

In the present work, we derive other properties of the class $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$ which are modified Hadamard product property, inclusion theorem, radius of close-to-convexity, star-likeness and convexity properties.

2. PRELIMINARIES

Lemma 2.1[6]: Let the function $g(z)$ be as defined by (1.2). Then $g(z) \in T_p^*(\alpha, \lambda)$ if and only

$$\sum_{k=1}^{\infty} (p+k)^\lambda (p+k-\alpha) b_{p+k} \leq p^\lambda (p-\alpha). \quad (2.1)$$

Now we prove the following Lemma which gives a sufficient condition for functions

$f(z)$ defined by (1.2) to be in the class $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$.

Lemma 2.2: Let the function $f(z)$ defined by (1.2) be in the class $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$. Then

$$\sum_{k=1}^{\infty} (1-\gamma B)(p+k)^{1+\lambda} a_{p+k} + [(\gamma B-1)p + (A-B(p-\beta)\gamma)] \frac{p^\lambda (p-\alpha)}{p+k-\alpha} \leq (A-B)\gamma p^\lambda (p-\beta). \quad (2.2)$$

Proof. Since $f \in T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$, there exists a function $g(z) \in T_p^*(\alpha, \lambda)$ such that

$$\left| \frac{z(D^\lambda f(z))' - pD^\lambda g(z)}{(A-B)(p-\beta)D^\lambda g(z) - B[z(D^\lambda f(z))' - pD^\lambda g(z)]} \right| < \gamma, \quad z \in U.$$

Then, we have

$$\begin{aligned} & \left| \frac{z(D^\lambda f(z))' - pD^\lambda g(z)}{(A-B)(p-\beta)D^\lambda g(z) - B[z(D^\lambda f(z))' - pD^\lambda g(z)]} \right| \\ &= \left| \frac{\sum_{k=1}^{\infty} (p+k)^{1+\lambda} a_{p+k} z^{p+k} - \sum_{k=1}^{\infty} p(p+k)^\lambda b_{p+k} z^{p+k}}{(A-B)(p-\beta)p^\lambda z^p + \sum_{k=1}^{\infty} B(p+k)^{1+\lambda} a_{p+k} z^{p+k} - \sum_{k=1}^{\infty} Bp(p+k)^\lambda b_{p+k} z^{p+k} - \sum_{k=1}^{\infty} (A-B)(p-\beta)(p+k)^\lambda b_{p+k} z^{p+k}} \right| < \gamma \end{aligned}$$

Since $\operatorname{Re}(z) \leq |z| \forall z$, we obtain

$$\operatorname{Re} \left\{ \frac{\sum_{k=1}^{\infty} (p+k)^{1+\lambda} a_{p+k} z^{p+k} - \sum_{k=1}^{\infty} p(p+k)^\lambda b_{p+k} z^{p+k}}{(A-B)(p-\beta)p^\lambda z^p + \sum_{k=1}^{\infty} B(p+k)^{1+\lambda} a_{p+k} z^{p+k} - \sum_{k=1}^{\infty} Bp(p+k)^\lambda b_{p+k} z^{p+k} - \sum_{k=1}^{\infty} (A-B)(p-\beta)(p+k)^\lambda b_{p+k} z^{p+k}} \right\} < \gamma. \quad (2.3)$$

We choose values of z in the real axis so that $\frac{z(D^\lambda f(z))'}{D^\lambda g(z)}$ is real.

Then upon simplifying the denominator in (2.3) and letting $z \rightarrow 1^-$ through real values, we obtain

$$\begin{aligned} & \left. \sum_{k=1}^{\infty} (p+k)^{1+\lambda} a_{p+k} - \sum_{k=1}^{\infty} p(p+k)^\lambda b_{p+k} \leq (A-B)p^\lambda(p-\beta) + \sum_{k=1}^{\infty} B\gamma(p+k)^{1+\lambda} a_{p+k} - \left\{ \sum_{k=1}^{\infty} [\gamma Bp(p+k)^\lambda + \gamma(A-B)(p-\beta)(p+k)^\lambda] b_{p+k} \right\} \right\} \\ \text{i.e. } & \sum_{k=1}^{\infty} (1-\gamma B)(p+k)^{1+\lambda} a_{p+k} + [(\gamma B-1)p + (A-B)(p-\beta)\gamma](p+k)^\lambda b_{p+k} \leq (A-B)p^\lambda(p-\beta). \end{aligned} \quad (2.4)$$

But from Lemma 2.1, $T_p^*(\alpha, \beta)$ implies

$$\sum_{k=1}^{\infty} (p+k)^\lambda b_{p+k} \leq \frac{p^\lambda(p-\alpha)}{p+k-\alpha} \quad (2.5)$$

So that by (2.4) and (2.5) we have

$$\sum_{k=1}^{\infty} (1-\gamma B)(p+k)^{1+\lambda} a_{p+k} + [(\gamma B-1)p + (A-B)(p-\beta)\gamma] \frac{p^\lambda(p-\alpha)}{p+k-\alpha} \leq (A-B)p^\lambda(p-\beta). \quad (2.6)$$

End of proof.

The result is sharp, the extremal function being

$$f(z) = z^p - \frac{(A-B)\gamma p k(p-\beta) - p^{1+\lambda}(\gamma B-1)(p-\alpha)}{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha)} z^{p+k}, \quad p, k \in \mathfrak{N} \quad (2.7)$$

for $k \geq 1$, with respect to

$$g(z) = z^p - \frac{p^\lambda(p-\alpha)}{(p+k)^\lambda(p+k-\alpha)} z^{p+k}.$$

Corollary 2.3. Let the function defined by (1.2) be in the class $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$. Then

$$a_{p+k} \leq \frac{(A-B)\gamma p k(p-\beta) + p^{1+\lambda}(1-\gamma B)(p-\alpha)}{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha)}, \quad \text{for } p, k \in \mathfrak{N}.$$

Remark 2. (i) The result obtained in Lemma 2.2 correct the result obtained by Güney (Theorem 2.2 of [12]), (ii) The result obtained in Corollary 2.3 also correct the result obtained by Güney (Corollary 2.3 of [12]).

3.0 Main results

In this section we provide the main results.

Our next result is the Quasi-Hadamard (Modified Hadamard product) for the class $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$.

3.1 Modified Hadamard Product

Let the functions $f_i(z)$ ($i = 1, 2$) be defined by

$$f_i(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,i} z^{p+k}, \quad p \in \mathfrak{K}. \quad (2.8)$$

Following the definition (1.5), the modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{k=1}^{\infty} a_{p+k,1} a_{p+k,2} z^{p+k}.$$

Theorem 3.1. Let the functions $f_i(z)$ ($i = 1, 2$) defined by (2.8) be in the class $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$. Then

$$(f_1 * f_2)(z) \in T_p^*(g(z), \delta, \beta, \gamma, A, B, \lambda), \text{ where}$$

$$\delta = \frac{(A-B)\mathcal{P}^\lambda(p-\beta) - (\mathcal{B}-1)p^{\lambda+2}[(1-\mathcal{B})(p+1)^{1+\lambda}(1+p-\alpha_1)][(1-\mathcal{B})(p+1)^{1+\lambda}(1+p-\alpha_2)]}{(\mathcal{B}-1)(p+1)^{1+\lambda}[(A-B)\mathcal{P}^\lambda(p-\beta) - (\mathcal{B}-1)p^{\lambda+1}(p-\alpha_1)][(A-B)\mathcal{P}^\lambda(p-\beta) - (\mathcal{B}-1)p^{\lambda+1}(p-\alpha_2)]} \\ - (\mathcal{B}-1)p^{1+\lambda}[(1-\mathcal{B})(p+1)^{1+\lambda}(1+p-\alpha_1)][(1-\mathcal{B})(p+1)^{1+\lambda}(1+p-\alpha_2)]} \quad (2.9)$$

The result is sharp for the functions $f_i(z)$ ($i = 1, 2$) given by

$$f_i(z) = z^p - \frac{(A-B)\mathcal{P}^\lambda(p-\beta) - (\mathcal{B}-1)p^{1+\lambda}(p-\alpha_i)}{(1-\mathcal{B})(p+1)^{1+\lambda}(1+p-\alpha_i)} z^{p+1}, \quad p \in \mathfrak{K}; \quad i = 1, 2. \quad (2.10)$$

Proof. Let $f_i(z) \in T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$ ($i = 1, 2$), we need to find the largest δ such that

$$\sum_{k=1}^{\infty} \frac{(1-\mathcal{B})(p+k)^{1+\lambda}(p+k-\delta)}{(A-B)\mathcal{P}^\lambda k(p-\beta) - (\mathcal{B}-1)p^{1+\lambda}(p-\delta)} a_{p+k,1} a_{p+k,2} \leq 1$$

Since $f_i(z) \in T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$ ($i = 1, 2$), in view of (2.2) we have

$$\sum_{k=1}^{\infty} \frac{(1-\mathcal{B})(p+k)^{1+\lambda}(p+k-\alpha_i)}{(A-B)\mathcal{P}^\lambda k(p-\beta) - (\mathcal{B}-1)p^{1+\lambda}(p-\alpha_i)} a_{p+k,i} \leq 1 \quad (i = 1, 2).$$

Furthermore, by the Cauchy-Schwarz inequalities, we get

$$\sum_{k=1}^{\infty} \frac{\sqrt{[(1-\mathcal{B})(p+k)^{1+\lambda}(p+k-\alpha_1)][(1-\mathcal{B})(p+k)^{1+\lambda}(p+k-\alpha_2)]}}{\sqrt{[(A-B)\mathcal{P}^\lambda k(p-\beta) - (\mathcal{B}-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\mathcal{P}^\lambda k(p-\beta) - (\mathcal{B}-1)p^{1+\lambda}(p-\alpha_2)]}} \sqrt{a_{p+k,1} a_{p+k,2}} \leq 1.$$

Thus it is sufficient to show that

$$\frac{(1-\mathcal{B})(p+k)^{1+\lambda}(p+k-\delta)}{(A-B)\mathcal{P}^\lambda k(p-\beta) - (\mathcal{B}-1)p^{1+\lambda}(p-\delta)} a_{p+k,1} a_{p+k,2} \\ \leq \frac{\sqrt{[(1-\mathcal{B})(p+k)^{1+\lambda}(p+k-\alpha_1)][(1-\mathcal{B})(p+k)^{1+\lambda}(p+k-\alpha_2)]}}{\sqrt{[(A-B)\mathcal{P}^\lambda k(p-\beta) - (\mathcal{B}-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\mathcal{P}^\lambda k(p-\beta) - (\mathcal{B}-1)p^{1+\lambda}(p-\alpha_2)]}} \sqrt{a_{p+k,1} a_{p+k,2}} \quad (k \in \mathfrak{K}),$$

Or equivalently,

$$\sqrt{a_{p+k,1} a_{p+k,2}} \\ \leq \frac{(A-B)\mathcal{P}^\lambda k(p-\beta) - (\mathcal{B}-1)p^{1+\lambda}(p-\delta) \sqrt{[(1-\mathcal{B})(p+k)^{1+\lambda}(p+k-\alpha_1)][(1-\mathcal{B})(p+k)^{1+\lambda}(p+k-\alpha_2)]}}{(1-\mathcal{B})(p+k)^{1+\lambda}(p+k-\delta) \sqrt{[(A-B)\mathcal{P}^\lambda k(p-\beta) - (\mathcal{B}-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\mathcal{P}^\lambda k(p-\beta) - (\mathcal{B}-1)p^{1+\lambda}(p-\alpha_2)]}} \quad (k \in \mathfrak{K}).$$

By noting that,

$$\sqrt{a_{p+k,1} a_{p+k,2}} \leq \frac{\sqrt{[(A-B)\mathcal{P}^\lambda k(p-\beta) - (\mathcal{B}-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\mathcal{P}^\lambda k(p-\beta) - (\mathcal{B}-1)p^{1+\lambda}(p-\alpha_2)]}}{[(1-\mathcal{B})(p+k)^{1+\lambda}(1+p-\alpha_1)][(1-\mathcal{B})(p+k)^{1+\lambda}(1+p-\alpha_2)]} \quad (k \in \mathfrak{K})$$

Consequently, we need only to prove that

$$\frac{(A-B)\gamma p^\lambda k(p-\beta) - (\gamma B-1)p^{1+\lambda}(p-\delta)\sqrt{[(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_1)][(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_2)]}}{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\delta)\sqrt{[(A-B)\gamma p^\lambda k(p-\beta) - (\gamma B-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\gamma p^\lambda k(p-\beta) - (\gamma B-1)p^{1+\lambda}(p-\alpha_2)]}}$$

$$\leq \frac{(A-B)\gamma p^\lambda k(p-\beta) - (\gamma B-1)p^{1+\lambda}(p-\delta)}{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\delta)} \quad (k \in \mathfrak{N}),$$

which is equivalent to

$$(1-\gamma B)(p+k)^{1+\lambda}(p+k-\delta)[(A-B)\gamma p^\lambda k(p-\beta) - (\gamma B-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\gamma p^\lambda k(p-\beta) - (\gamma B-1)p^{1+\lambda}(p-\alpha_2)]$$

$$\leq (A-B)\gamma p^\lambda k(p-\beta) - (\gamma B-1)p^{1+\lambda}(p-\delta)[(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_1)][(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_2)]$$

Such that

$$\delta\{(1-\gamma B)\gamma p^\lambda k(p+k)^{1+\lambda}[(A-B)\gamma p^\lambda k(p-\beta) - (\gamma B-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\gamma p^\lambda k(p-\beta) - (\gamma B-1)p^{1+\lambda}(p-\alpha_2)]$$

$$- (1-\gamma B)p^{1+\lambda}[(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_1)][(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_2)]\}$$

$$\leq (A-B)\gamma p^\lambda k(p-\beta) - (\gamma B-1)p^{1+\lambda}[(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_1)][(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha_2)]$$

$$- (1-\gamma B)(p+k)^{1+\lambda}[(A-B)\gamma p^\lambda k(p-\beta) - (\gamma B-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\gamma p^\lambda k(p-\beta) - (\gamma B-1)p^{1+\lambda}(p-\alpha_2)]$$

i.e.

$$(A-B)\gamma p^\lambda k(p-\beta) - (\gamma B-1)p^{1+\lambda}[(1-\gamma B)(p+k)^{1+\lambda}(k+p-\alpha_1)][(1-\gamma B)(p+k)^{1+\lambda}(k+p-\alpha_2)]$$

$$\delta = \frac{- (1-\gamma B)(p+1)^{1+\lambda}[(A-B)\gamma p^\lambda k(p-\beta) - (\gamma B-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\gamma p^\lambda k(p-\beta) - (\gamma B-1)p^{1+\lambda}(p-\alpha_2)]}{(\gamma B-1)(p+k)^{1+\lambda}[(A-B)\gamma p^\lambda k(p-\beta) - (\gamma B-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\gamma p^\lambda k(p-\beta) - (\gamma B-1)p^{1+\lambda}(p-\alpha_2)]} = E(k) \quad (k \in \mathfrak{N}) \quad (2.11)$$

$$- (\gamma B-1)p^{1+\lambda}[(1-\gamma B)(p+k)^{1+\lambda}(k+p-\alpha_1)][(1-\gamma B)(p+k)^{1+\lambda}(k+p-\alpha_2)]$$

Since $E(k)$ is an increasing function of k ($k \in \mathfrak{N}$), letting $k=1$, we obtain

$$(A-B)\gamma p^\lambda (p-\beta) - (\gamma B-1)p^{1+\lambda}[(1-\gamma B)(p+1)^{1+\lambda}(1+p-\alpha_1)][(1-\gamma B)(p+1)^{1+\lambda}(1+p-\alpha_2)]$$

$$\delta = \frac{- (1-\gamma B)(p+1)^{1+\lambda}[(A-B)\gamma p^\lambda (p-\beta) - (\gamma B-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\gamma p^\lambda (p-\beta) - (\gamma B-1)p^{1+\lambda}(p-\alpha_2)]}{(\gamma B-1)(p+1)^{1+\lambda}[(A-B)\gamma p^\lambda (p-\beta) - (\gamma B-1)p^{1+\lambda}(p-\alpha_1)][(A-B)\gamma p^\lambda (p-\beta) - (\gamma B-1)p^{1+\lambda}(p-\alpha_2)]}$$

$$- (\gamma B-1)p^{1+\lambda}[(1-\gamma B)(p+1)^{1+\lambda}(1+p-\alpha_1)][(1-\gamma B)(p+1)^{1+\lambda}(1+p-\alpha_2)]$$

which is (2.9) that is required.

Corollary 3.2. For $f_i(z)$ ($i=1,2$) as in Theorem 3.1, we have

$$h(z) = z^p - \sum_{k=1}^{\infty} \sqrt{a_{p+k,1} a_{p+k,2}} z^{p+k}$$

belongs to the class $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$ ($i=1,2$). The result is sharp with the function given by (2.10).

Proof. The result follows from the Cauchy-Schwarz Inequality.

3.2 INCLUSION PROPERTIES

Theorem 3.2. Let the functions $f_i(z)$ ($i=1,2$) defined by (2.8) be in the class $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$ ($i=1,2$).

Then the function

$$F(h) = z^p - \sum_{k=1}^{\infty} (a_{p+k,1}^2 + a_{p+k,2}^2) z^{p+k} \text{ belongs to } T_p^*(g(z), \delta, \beta, \gamma, A, B, \lambda) \text{ (} i=1,2 \text{), where}$$

$$\delta = \frac{(1-\gamma B)(p+1)^2(p+1-\alpha)^2[(A-B)\gamma p(p-\alpha) + (1-\gamma B)p^{1+\lambda}] - 2(p+1)[(A-B)(p-\alpha)\gamma p + p^{1+\lambda}(1-\gamma B)(p-\alpha)]^2}{p^{1+\lambda}(1-\gamma B)^2(1+p-\alpha)^2 - 2[(A-B)\gamma p(p-\alpha) + p^{1+\lambda}(1-\gamma B)(p-\alpha)]^2}$$

The result is sharp for the function $f_i(z)$ ($i=1,2$) defined by (2.10).

Proof. By virtue of Lemma 2.2, we obtain

$$\sum_{k=1}^{\infty} \left\{ \frac{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha)}{(A-B)(p-\alpha)\gamma p k - p^{1+\lambda}(\gamma B-1)(p-\alpha)} \right\}^2 a_{p+k,1}^2 \leq \left\{ \sum_{k=1}^{\infty} \frac{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha)}{(A-B)(p-\alpha)\gamma p k - p^{1+\lambda}(\gamma B-1)(p-\alpha)} a_{p+k,1} \right\}^2 \leq 1 \quad (2.12)$$

and

$$\sum_{k=1}^{\infty} \left\{ \frac{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha)}{(A-B)(p-\alpha)\gamma p k - p^{1+\lambda}(\gamma B-1)(p-\alpha)} \right\}^2 a_{p+k,2}^2 \leq \left\{ \sum_{k=1}^{\infty} \frac{(1-\gamma B)(p+k)^{1+\lambda}(p+k-\alpha)}{(A-B)(p-\alpha)\gamma p k - p^{1+\lambda}(\gamma B-1)(p-\alpha)} a_{p+k,2} \right\}^2 \leq 1. \quad (2.13)$$

It follows from (2.12) and (2.13) that

$$\sum_{k=1}^{\infty} \frac{1}{2} \left\{ \frac{(1-\gamma\mathcal{B})(p+k)^{1+\lambda}(p+k-\alpha)}{(A-B)(p-\alpha)\gamma pk - p^{1+\lambda}(\gamma\mathcal{B}-1)(p-\alpha)} \right\}^2 (a_{p+k,1}^2 + a_{p+k,2}^2) \leq 1.$$

Therefore, we need to find the largest δ such that

$$\frac{(1-\gamma\mathcal{B})(p+k)^{1+\lambda}(p+k-\delta)}{(A-B)(p-\alpha)\gamma pk - p^{1+\lambda}(\gamma\mathcal{B}-1)(p-\delta)} \leq \frac{1}{2} \left\{ \frac{(1-\gamma\mathcal{B})(p+k)^{1+\lambda}(p+k-\alpha)}{(A-B)(p-\alpha)\gamma pk - p^{1+\lambda}(\gamma\mathcal{B}-1)(p-\alpha)} \right\}^2, \quad k \geq 1 \quad (2.14)$$

A simple computation yields that

$$\delta \leq \frac{(1-\gamma\mathcal{B})(p+k)^2(p+k-\alpha)^2[(A-B)\gamma pk(p-\alpha) + (1-\gamma\mathcal{B})p^{1+\lambda}] - 2(p+k)[(A-B)(p-\alpha)\gamma pk + p^{1+\lambda}(1-\gamma\mathcal{B})(p-\alpha)]^2}{p^{1+\lambda}(1-\gamma\mathcal{B})^2(p+k)^2(p+k-\alpha)^2 - 2[(A-B)(p-\alpha)\gamma pk + p^{1+\lambda}(1-\gamma\mathcal{B})(p-\alpha)]^2}$$

$= G(k), \quad k \in \mathfrak{N}$

Since $G(k)$, is an increasing function of $k(k \geq 1)$ for the parameters involve and $p \in \mathfrak{N}$. letting $k=1$, we have

$$\delta = \frac{(1-\gamma\mathcal{B})(p+1)^2(p+1-\alpha)^2[(A-B)\gamma p(p-\alpha) + (1-\gamma\mathcal{B})p^{1+\lambda}] - 2(p+1)[(A-B)(p-\alpha)\gamma p + p^{1+\lambda}(1-\gamma\mathcal{B})(p-\alpha)]^2}{p^{1+\lambda}(1-\gamma\mathcal{B})^2(1+p-\alpha)^2 - 2[(A-B)\gamma p(p-\alpha) + p^{1+\lambda}(1-\gamma\mathcal{B})(p-\alpha)]^2},$$

which completes the proof of the theorem.

3.3 RADII OF CLOSE-TO-CONVEX, STARLIKENESS AND CONVEXITY

Theorem 3.3. Let the function $f(z)$ defined by (1.2) be in the class $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$ ($i=1, 2$).

Then $f(z)$ is P-valently close-to-convex of order $\rho(0 \leq \rho < p)$ in $|z| < r_1$ where

$$r_1 = \inf_{k \in \mathfrak{N}} \left\{ \frac{(p-\delta)(1-\gamma\mathcal{B})(p+k)^{1+\lambda}(p+k-\alpha)}{(p+k)[(A-B)\gamma pk(p-\beta) + p^{1+\lambda}(1-\gamma\mathcal{B})(p-\alpha)]} \right\}^{\frac{1}{k}}.$$

The result is sharp with the extremal function given by (2.10).

Proof. It suffices to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \rho, \quad |z| < r_1. \quad \text{In deed we have}$$

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq \sum_{k=1}^{\infty} (p+k) a_{p+k} |z|^k \quad (2.15)$$

Hence (2.15) is true if

$$\sum_{k=1}^{\infty} (p+k) a_{p+k} |z|^k \leq p - \rho \quad \text{or}$$

$$\sum_{k=1}^{\infty} \frac{(p+k)}{(p-\rho)} a_{p+k} |z|^k \leq 1 \quad (2.16)$$

By the Theorem, (2.15) is true if

$$\frac{(p-\delta)|z|^k}{p-\rho} \leq \frac{(p-\delta)(1-\gamma\mathcal{B})(p+k)^{1+\lambda}(p+k-\alpha)}{(p+k)[(A-B)\gamma pk(p-\beta) + p^{1+\lambda}(1-\gamma\mathcal{B})(p-\alpha)]}, \quad k \geq 1 \quad (2.17)$$

Solving (2.17) for $|z|$, we get the desired result. That is,

$$r_1 = \inf_{k \in \mathfrak{N}} \left\{ \frac{(p-\delta)(1-\gamma\mathcal{B})(p+k)^{1+\lambda}(p+k-\alpha)}{(p+k)[(A-B)\gamma pk(p-\beta) + p^{1+\lambda}(1-\gamma\mathcal{B})(p-\alpha)]} \right\}^{\frac{1}{k}}.$$

Theorem 3.4. Let the function $f(z)$ defined by (1.2) be in the class $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$ ($i=1, 2$).

Then $f(z)$ is P-valently starlike of order $\rho(0 \leq \rho < p)$ in $|z| < r_2$ where

$$r_2 = \inf_{k \in \mathfrak{N}} \left\{ \frac{(p-\delta)(1-\gamma\mathcal{B})(p+k)^{1+\lambda}(p+k-\alpha)}{(p+k-\rho)[(A-B)\gamma pk(p-\beta) + p^{1+\lambda}(1-\gamma\mathcal{B})(p-\alpha)]} \right\}^{\frac{1}{k}}.$$

The result is sharp with the extremal function given by (2.10).

Theorem 3.5. Let the function $f(z)$ defined by (1.2) be in the class $T_p^*(g(z), \alpha, \beta, \gamma, A, B, \lambda)$ ($i = 1, 2$).

Then $f(z)$ is P-valently convex of order ρ ($0 \leq \rho < p$) in $|z| < r_3$ where

$$r_3 = \inf_{k \in \mathbb{N}} \left\{ \frac{p(p-\rho)(1-\gamma\mathcal{B})(p+k)^{1+\lambda}(p+k-\alpha)}{(p+k)(p+k-\rho)[(A-B)\gamma k(p-\beta) + p^{1+\lambda}(1-\gamma\mathcal{B})(p-\alpha)]} \right\}^{\frac{1}{k}}.$$

The result is sharp with the extremal function given by (2.10).

References

- [1] Goodman A. W., *Univalent Function*, Mariner Publishing Company, Inc. USA, 1983.
- [2] Aouf, M.K., "A generalization of multivalent functions with negative coefficients," *J. Korean Math. Soc.* 25(1), 53-66, 1988.
- [3] Hossen, H. M., "Quasi-Hadamard product of certain p-valent functions, *Demonstration Math.* 33(2), 277-281, 2000.
- [4] Khairnar, S. M. and More, N. H., "On a subclass of certain p-valent starlike functions with negative coefficients," *General Mathematics*, vol. 18, No. 9, 95-119, 2010.
- [5] Sharma, P. and Srivastava, P., "Some properties of a class of p-valent analytic functions associated with convolution," *European Journal of Pure and Applied Mathematics*, vol. 3, No. 6, 1093-1112, 2010.
- [6] Amsheri, S. M., Zharkova, V., "Subclasses of p-valent starlike functions defined by using certain fractional derivative operator," *International Journal of Mathematical Science Education*, vol. 4, No. 1, 17-32, 2011.
- [7] Amsheri, S. M., Zharkova, V., "Subclasses of p-valent functions defined by convolution involving certain fractional derivative operator," *Int. j. Contemp. Math. Sciences*, vol. 6, No. 32, 1545-1567, 2011.
- [8] Pathak, M. and Sharma P., "A family of p-valent functions defined by a fractional calculus operator," *Int. J. of Pure and Applied Mathematics* vol. 87, No. 1, 129-143, 2013.
- [9] Jeyaraman, M. P., Suresh, T. K. and Reddy, K E., "Some subclasses of p-valent Analytic Functions," *Global Journal. Of Science Frontier and Decision Sciences*, vol. 13, Issue 3 Version 1.0, 2013.
- [10] Arif, M., Sokol, J. and Ayaz, M., "Coefficient Inequalities for a of p-valent Analytic functions," *The Scientific World Journal*, vol. 2014, Article ID801751, 5 pages.
- [11] Noor, K. I. and Khan, N., "Some Classes of p-valent Analytic functions Associated with Hypergeometric Functions," *Filomat* 29: 5(2015), 1031-1038, DOI 10.2298/FIL 1505031N. Available at <http://www.pmf.ni.ac.rs/filomat>.
- [12] Güneş, H. Ö. "A new Class of Salagean-Type P-valent functions," *Advances in inequalities for Series*, 121-129, 2008.
- [13] Acu, M. "On Some Analytic Functions with negative coefficients," *General Mathematics* Vol. 15, Nr. 2-3, 190-200. 2007.
- [14] Salagean, G.S, *Subclasses of Univalent Functions: Lecture Notes in Mathematics*, Springer-Verlag Publisher, Heidelberg and Newyork, 1983, 362-372.