Analytical Solution of Risk Adjusted Option Pricing Model By Variational Iteration method

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Abstract:The work presents analytical solution of a non-linear Black Scholes-equation (Option Pricing Model) with transaction cost measure and volatile portfolio risk measure. The analytical solution was obtained by using variational iteration method.In this method the problems are initially approximated with possible unknowns, and then a correction functional is constructed by a general Lagrange multiplier which can be identified optimally via the variational theory.Under some given conditions, we obtain similar solution as for the linear counterpart found in literature.

Keywords:Black-Scholes equation,Variational Iteration method, Lagrange multiplier

1. Introduction

The pricing of options is a central problem in financial market.Options are widely used on markets and exchanges. In option pricing theorem the Back-Scholes equation [1] is one of the most effective models for pricing options. The equation assumes the existence of perfect capital markets and the security prices are log normally distributed or,equivalently,the log-returns are normally distributed.to this one adds the assumptions that trading in all security is continuous and that the distribution of the rates of return is stationary [2,3,4,5].

Transaction costs as well as the volatile portfolio risk depend on the time –lag between two consecutive transactions.Minimizing their sum yields the optimal length of the hedge interval-time lag. This leads to a fully non-linear parabolic Partial Differential Equation.If transaction costs are taken into account perfect replication of the contingent claim is no longer possible. Therisk adjusted option pricing model can be viewed as an equation with a variable volatility coefficient [6]. The nonlinear parabolicpartial differential equation with transaction cost and volatile portfolio risk measure is written as

$$\partial_t V + \frac{\hat{\sigma}^2(x,t)}{2} x^2 u_{xx} + (r-\delta) x u_x - ru = (r_{TC} + r_{VP}) x$$

where

$r_{TC} = \frac{C|\Gamma|\hat{\sigma}x}{\sqrt{2\pi}} \frac{1}{\sqrt{\Delta t}}$is the transaction costs measure

$r_{Vp} = \frac{1}{2} R \hat{\sigma}^4 x^2 \Gamma^2 \Delta t$ is the volatile portfolio risk measure

And $\Gamma = u_{xx}$

They change in the value of the portfolio after minimizing the total risk with respect to time lag [8] is given as

$$u_t + \frac{\sigma^2(1-\mu(S\partial_s^2 v(s,t))^{\frac{1}{3}}}{2} x^2 u_{xx} + (r-\delta) x u_x - ru - Ax^2 u_{xx} = 0$$

$\Rightarrow u_t + \frac{x^2}{2} \hat{\sigma}^2 u_{xx} + (r-\delta) x u_x - ru - Ax^2 u_{xx} = 0,$ (1)

where

$A = (\frac{3}{2})^{\frac{3}{2}} \left(\frac{C^2 R}{2\pi}\right)^{\frac{1}{2}} \sigma^3, \hat{\sigma}^2(x,t) = \sigma^2(1-\mu\left(x u_{xx}(x,t)\right)^{\frac{1}{3}},$ (2)

$r$is the risk free rate, $\delta\ is\ the\ dividend\ yield$, σ is the volatility.$x$is the stock price where $\hat{\sigma}^2(x,t)$ depends on a solution $u(x,t)\ and\ \mu = 3\left(\frac{C^2 R}{2\pi}\right)^{\frac{1}{3}}$.

In this work we investigate equation (1) with a view to obtaining the analytical solution to the terminal condition

$u(x,t) = g(x)$. (3)

By using the variational iterative method (VIM), we assume throughout that g has derivatives of all orders.An advantage of the variational iteration method is that it can be applied directly for all types of differential and integral equations,homogenous or inhomogeneous[2,7,8,1]. The method is capable of greatly reducing the size of computational work while still maintaining high accuracy of the numerical solution.

2. VARIATIONAL ITERATION METHOD

In this method, the solution of a differential equation with a linearization assumption is used as an initial approximation, and then a more precise approximation at some special point can be obtained. To illustrate this method, consider the differential equation in the formal form

$$Lu(x,t) + Nu(x,t) = g(x,t),$$

where$L$ is a linear operator,N is a nonlinear operator ,and $g(x,t)$ is an inhomogeneous term.

According to the VIM, we can construct a correctional functional as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(Lu_n(x,s) + N\acute{u}_n(x,s) - g(x,s)) \,, \qquad (4)$$

where λ is a general Lagrangian multiplier [9,10] ,and can be identified optimally via the variational theory,the subscript n denotes the $n-th$ order approximation,$\acute{u}_n$is considered as a restricted variation [9,10] ,i.e.$\delta\acute{u}_n = 0$. Equation(4) is called a correction functional.The successive approximations $u_{n+1}, n \geq 0$,of the solution $u$will be readily obtained by suitable choice of trial function u_0. Consequently ,the solution is given as

$$u(x,t) \lim_{n\to\infty} u_n(x,t) \qquad (5)$$

3. APPLICATION

In this section, we illustrate the proposed method to risk adjusted Black Scholes model transaction cost measure and volatile portfolio risk measure.

According to the VIM,we construct a correction functional for(1) in the form

$$u_{n+1}(x,t) = u_n(x,t) + \int_t^T \lambda\left((u_n)_s + \frac{\sigma^2}{2}x^2(\acute{u}_n)_{xx} + (r-\delta)x(\acute{u}_n)_x - r(\acute{u}_n) - Ax^2(\acute{u}_n)_{xx}\right) ds, \qquad (6)$$

whereλ is a general Lagrange multiplier, and $\acute{u}$ denotes the restricted variation ie$\delta\acute{u}$=0.

The correction functional (6)

$$
\begin{aligned}
\delta u_{n+1}(x,t) \\
&= \delta u_n(x,t) \\
&\quad + \delta \int_t^T \lambda\Big((u_n)_s + \frac{\sigma^2}{2} x^2 (\acute{u}_n)_{xx} + (r-\delta) x (\acute{u}_n)_x - r(\acute{u}_n) \\
&\quad - A x^2 (\acute{u}_n)_{xx} \Big) ds
\end{aligned}
$$

$$
= \delta u_n(x,t) + \int_t^T \lambda\,(u_n)_s ds
$$

$$
= \delta u_n(x,t) - \lambda \delta u_n(x,s) I_{s=t} - \int_t^T \lambda'\, \delta u_n(x,s) = 0,
$$

yields the following stationary condition

$$
-\lambda'(s) = 0,
$$

$$
1 - \lambda(s) I_{s=t} = 0.
$$

The Lagrange multiplier can be identified as

$$
\lambda = -1.
$$

Substituting this value of Lagrange multiplier into the functional (6) gives the iteration formula.

$$
\begin{aligned}
u_{n+1}(x,t) &= u_n(x,t) \\
&\quad + \int_t^T \Big((u_n)_s + \frac{\sigma^2}{2} x^2 (u_n)_{xx} + (r-\delta) x (u_n)_x - r(u_n) - A x^2 (u_n)_{xx} \Big) ds \qquad (7)
\end{aligned}
$$

Where

$$
u_n(x,t) = \sum_{k=0}^{n} \left[\sum_{m=0}^{2k} \left\{ \sum_{v=0}^{m} \frac{(-1)^{(m-v)}}{v!\,(m-v)!} \left(\left(\frac{\sigma^2 v}{2} + r \right)(v-1) - \delta v \right)^k \right\} x^m g^{(m)}. \right] \frac{(T-t)^k}{k!},
$$

where$n \geq 0$.Then ,the exact solution is given as series

$u(x,t) = lim_{n\to\infty} u_n(x,t) =$
$\sum_{k=0}^{\infty}\left[\sum_{m=0}^{2k}\left\{\sum_{v=0}^{m}\frac{(-1)^{(m-v)}}{v!(m-v)!}\left(\left(\frac{\sigma^2 v}{2}+r\right)(v-1)-\delta v\right)^k\right\}x^m g^{(m)}(x).\right]\frac{(T-t)^k}{k!}$(8)

LEMMA 1: Let

$$\varphi_k(x) = \sum_{m=0}^{2k}\left\{\sum_{v=0}^{m}\frac{(-1)^{(m-v)}}{v!\,(m-v)!}\left(\left(\frac{\sigma^2 v}{2}+r\right)(v-1)-\delta v\right)^k\right\}x^m g^{(m)}(x).$$

Then the defined function $\varphi_k(x)$ satisfies the recursion

$$\frac{\sigma^2}{2}x^2\varphi_k{}''(x)+(r-\delta)x\varphi'{}_k(x)-r\varphi_k-Ax^2\varphi''{}_k=\varphi_{k+1}(x)\;for\;all\;k\in\mathbb{N}_0 \qquad (9)$$

Proof.

For convenience we introduce the notation

$$\gamma_m^{(k)} = \sum_{v=0}^{m}\frac{(-1)^{(m-v)}}{v!\,(m-v)!}\left(\left(\frac{\sigma^2 v}{2}+r\right)(v-1)-\delta v\right)^k.$$

Then φ_k can be written as

$$\varphi_k(x) = \sum_{m=0}^{2k}\gamma_m^{(k)}\,x^m g^{(m)}(x)\;for\;all\;k\epsilon\mathbb{N}_0.$$

Now

$$\varphi_k'(x) = \sum_{m=0}^{2k}\gamma_m^{(k)}\left[mx^{m-1}g^{(m)}(x)+x^m g^{(m+1)}(x)\right]$$

$$\varphi_k''(x) = \sum_{m=0}^{2k}\gamma_m^{(k)}\left[m(m-1)x^{m-2}g^{(m)}(x)+2mx^{m-1}g^{(m+1)}(x)+x^m g^{(m+2)}(x)\right]$$

So that

$$\frac{\sigma^2}{2}x^2\varphi_k{}''(x)+(r-\delta)x\varphi'{}_k(x)-r\varphi_k-Ax^2\varphi''{}_k$$

$$= \sum_{m=0}^{2k} \gamma_m^{(k)} \left\{ \left[\frac{\sigma^2}{2} m(m-1) + (r-\delta)m - r - AM(m-1) \right] x^m g^{(m)}(x) \right.$$
$$\left. + \left[2\frac{\sigma^2}{2} m + (r-\delta) - 2Am \right] x^{m+1} g^{(m+1)}(x) + (\frac{\sigma^2}{2} + A) x^{m+2} g^{(m+2)}(x) \right\}$$

$$= \sum_{m=0}^{2k} \left[\frac{\sigma^2}{2} m(m-1) + (r-\delta)m - r - AM(m-1) \right] \gamma_m^{(k)} x^m g^{(m)}(x)$$
$$+ \sum_{m=1}^{2k+1} \left[2\frac{\sigma^2}{2} m + (r-\delta)m - 2Am \right] \gamma_{m-1}^{(k)} x^m g^{(m)}(x)$$
$$+ \sum_{m=2}^{2k+2} \left(\frac{\sigma^2}{2} + A \right) \gamma_{m-2}^{(k)} x^m g^{(m)}(x) = \sum_{m=0}^{2k+2} \gamma_m^{(k+1)} x^m g^{(m)}(x) = \varphi_{k+1}(x)$$

THEOREM 1: Letthe function $u(x,t)$be defined by

$$u(x,t) = \sum_{k=0}^{\infty} \varphi_k(x) \frac{(T-t)^k}{k!}. \qquad (10)$$

Then $u(x,t)$ satisfies equation (1).

PROOF.Substituting $u(x,t)$ into equation (1) gives

$$u_t + \frac{\sigma^2}{2} x^2 u_{xx} + (r-\delta) x u_x - ru - Ax^2 u_{xx}$$

$$= \sum_{k=0}^{\infty} \varphi_k(x) \frac{-k(T-t)^{(k-1)}}{k!} + \sum_{k=0}^{\infty} \frac{\sigma^2}{2} x^2 \varphi_k''(x) \frac{(T-t)^k}{k!} + \sum_{k=0}^{\infty} (r-\delta) x \varphi_k'(x) \frac{(T-t)^k}{k!}$$
$$- \sum_{k=0}^{\infty} r\varphi_k(x) \frac{(T-t)^k}{k!} - \sum_{k=0}^{\infty} Ax^2 \varphi_k''(x) \frac{(T-t)^k}{k!}$$

$$- \sum_{k=1}^{\infty} \varphi_k(x) \frac{(T-t)^{(k-1)}}{(k-1)!}$$
$$+ \sum_{k=0}^{\infty} \left(\frac{\sigma^2}{2} x^2 \varphi_k''(x) + (r-\delta) x \varphi_k'(x) - r\varphi_k(x) - Ax^2 \varphi_k''(x) \right) \frac{(T-t)^k}{k!}$$

$$= - \sum_{k=0}^{\infty} \varphi_k(x) \frac{(T-t)^{(k-1)}}{(k-1)!} + \sum_{k=1}^{\infty} \varphi_{k+1}(x) \frac{(T-t)^{(k)}}{(k-1)!}$$

$$= -\sum_{k=0}^{\infty} \varphi_k(x) \frac{(T-t)^{(k-1)}}{(k-1)!} + \sum_{k=1}^{\infty} \varphi_k(x) \frac{(T-t)^{(k-1)}}{(k-1)!}$$

$$= 0$$

4 Conclusion

We have appliedin this research work the Variational iteration method to obtaina theoretical analysis and closed form of the solution of Black-Scholes equation with transaction cost measure and volatile portfolio risk measure. Under some given conditions, we have obtained similar solution as for the linear counterpart as in [11].

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