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Reduced Differential Transform Method For Solving Partial Integro-Differential Equation

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Abstract

In this research work, Reduced Differential Transform Method (RDTM) is applied to solve both homogenous and non-homogeneous Partial Integro Differential Equation (PIDE).Partial integro differential equation (PIDE) has found wide applications in financial, biological, engineering and other problems in natural sciences. The proposed method to solve partial integro differential equation (PIDE) has been applied to variety of problems and have proved to yield highly accurate solution. Comparison is made between the exact solution and the result of reduced differential transformation method (RDTM) in order to verify the accuracy of the results, revealing the fact that this method is very effective and simple. This makes it preferable to some existing methods.

Keywords: Partial Integro Differential Equation (PIDE), Reduced Differential Transform Method (RDTM)

1. Introduction

Real life phenomena are often modeled by Ordinary/Partial Differential Equations. Due to the local nature of Ordinary Differential Operator (ODO), the models containing merely ODOs do not help in modeling memory and hereditary properties. One of the best remedies to overcome this drawback is the introduction of integral term in the model. The Ordinary/Partial Differential Equation along with the weighted integral of unknown function gives rise to an Integro-Differential Equation (IDE) or a Partial Integro-Differential Equation (PIDE) respectively. Analysis of such equations can be found in Existence and Uniqueness of a solution by the method of line [1].

Applications of PIDEs can be found in various fields.They occur naturally in various mathematical modeling of many fields: Physical, Biological, Engineering and Social Sciences in which it is necessary to take into account the effect of real world problems [2]. The suitability of the solution method for machine computation, and the inherent simplicity of the structure of the subject, combines to make the Partial Integro-Differential Equation approach a very valuable one for many applications.

Variational Iteration Method (VIM) has been used in [3] to solve PIDEs arising in heat conduction of materials with memory. Various numerical schemes are proposed in[4] to solve PIDEs arising in viscoelasticity. Nonlinear PIDEs arising in Nuclear Reactor Dynamics are solved in[5]. PIDEs have been used in Jump-Diffusion Models for pricing of derivatives in finance [6]. In[7] a nonlinear PIDE in financial modeling was used. In [8], a PIDE in the modeling of electricity swaptions was proposed. A PIDE governing bio-fluid flow in fractured biomaterials is proposed in[9]. The Laplace transform method have been used to solve linear non-homogeneous PDEs arising from water flow and heat transferred in fractured rocks [10]. In [11], a new method for nonlinear oscillatory systems using the Laplace transformation method was proposed. Similarly, Stiff systems of ODEs are solved in[12] using a combined Laplace Transformation (LT) and Homotophy Perturbation (HP) methods.In[13],a Laplace Transformation (LT) method was utilized to solve problems arising in fractional differential equations.

The advantage of the Partial Integro-Differential Equations representation for a variety of problem is witnessed by its increasing frequency in literatures and in many texts on method of advanced applied mathematics. This study is motivated by the less attention given the RTDM by earlier researchers on the approximate solution methods for PIDEs [14, 15, 16, 17].

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2.0 Basics Of Reduced Differential Transform Method (RDTM)

To illustrate the basics/basic concepts of RDTM, we consider the following general differential equation.

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = G(x,t) \qquad (1)$$

With respect to the Boundary Condition

$$u(0,t) = h(x);\ u_x(0,t) = f(x) \qquad (2)$$

Where $L = \frac{\partial}{\partial t}$ and $R = \frac{\partial^2}{\partial x^2}$ are linear operators with respects to function time (t) and distance (x) respectively; $Nu(x,t)$ is a Non-Linear term and $G(x,t)$ is the source term showing the homogenous or non-homogenous state of the system.

According to the RDTM we construct the following iteration formula:

$$Lu_k(t) + (k+1)(k+2)u_{k+2}(t) + Nu_k(t) = G(t) \qquad (3)$$

Where u_{k+2}, $Lu_k(t)$, $Nu_k(t)$ and $G(t)$are transformations of the functions $Ru(x,t), Lu(x,t), Nu(x,t)$ and $G(x,t)$ respectively.

From the Boundary Condition we write;

$$u_o(t) = h(t);\ u_1(t) = f(t) \qquad (4)$$

Substituting (3) into (4), we obtain;

When $k = 0$

$$Lu_0(t) + 2u_2(t) + Nu_0(t) = G(t)$$

$$2u_2(t) = G(t) - Lu_0(t) - Nu_0(t)$$

$$u_2(t) = \frac{1}{2}[G(t) - Lu_0(t) - N_0(t)]$$

When $k = 1, 2, \dots\dots\dots n$

$$u_3(t) = \frac{1}{2}[G(t) - Lu_1(t) - Nu_1(t)]$$

$$\vdots$$

$$u_n(t) = \frac{1}{n}!\,[G(t) - Lu_{n-2}(t) - Nu_{n-2}(t)]$$

Then;

The inverse transformation of the set of values $\{u_k(x)\}_{k=0}^{n}$ gives the approximation solution as;

$$\tilde{u}_n(x,t) = \sum_{k=0}^{n} u_k(t)x^k \qquad (5)$$

$$\text{i.e.}\ \tilde{u}_n(x,t) = \tilde{u}_0 + \tilde{u}_1 x + \tilde{u}_2 x^2 + \cdots + \tilde{u}_n x^n \qquad (6)$$

Where n is order of approximation solution.

The exact solution of the problem is given by;

$$u(x,t) = \lim_{n\to\infty} \bar{u}_n(x,t)$$

NOTE: The following is the transformation of some functional operator.

No.	FUNCTIONAL FORM	TRANSFORMED FORM
1	$u(x,t)$	$u_k(t) = \frac{1}{k!}[\frac{\delta k}{\delta x^k} u(x,t)]t = 0$
2	$W(x,t) = \{u(x,t) \pm v(x,t)$	$W_k(t) = u_k(t) \pm v_k(t)$
3	$W(x,t) = \propto u(x,t)$	$W_k(t) = \propto u_k(t)$ {$\propto$ is a constant}
4	$W(x,t) = x^m t^n$	$W(x,t) = t^n \delta(k-m) = t^n \begin{cases} 1 & k = m \\ 0 & k \neq m \end{cases}$
5	$W(x,t) = x^m t^n u(x,t)$	$W_k(t) = t^n u_{k-m}(t)$
6	$W(x,t) = u(x,t)v(x,t)$	$W_k(t) = \sum_{k=0}^{n} v_k(t)\, u_{n-k}(t) = \sum_{k=0}^{n} U_k(t)\, V_{n-k}(t)$
7	$W(x,t) = \frac{\partial^r}{\partial t^r} u(x,t)$	$W_k(t) = k + 1 \dots\dots (k+r)u_{k+r}(t)$ $= \frac{(k+1)!}{k!} u_{k+r}(t)$
8	$W(x,t) = \frac{\partial}{\partial x} u(x,t)$	$W_k(t) = \frac{\partial}{\partial x} u_k(t)$
9	$W(x,t) = e^{mn}$	$W_k(t) = (\frac{m}{k!})_{k\neq m}$

3.0 NUMERICAL EXAMPLES

Example 1

Considering the Equation;

$$u_{tt} = u_x + 2\int_0^t (t-s)\, u(x,s)ds - 2e^x \qquad (7)$$

With Boundary Condition $u(0,t) = \cos t$

Solving equation (7) by (RTDM) is as follows;

$$u_{tt} = u_x + 2\int_o^t (t-s)\left(u\,(x\,,s)\right)ds - 2e^x$$

$$(k+1)u_{k+1} = u_{ktt} + 2\left(\frac{1}{k!}\right) - 2\int_0^t [\,(t-s)u_k\,(x,s)]ds \qquad (8)$$

From boundary condition $U(0,t) = \cos t$

$U_o = \cos t$ (9)

When $k = 0$ in equation (8),

$$u_1 = u_{0tt} + 2\left(\frac{1}{0!}\right) - 2\int_0^t [\,(t-s)u_0\,(x\,,s)]ds$$

That is; $u_1 = \cos t$ (10)

When $k = 1$ in equation (8),

$$2u_2 = u_{1tt} + 2\left(\frac{1}{1!}\right) + 2\int_0^t [(t-s)\,u_1(x\,,s)]ds$$

That is; $u_2 = \frac{1}{2}\cos t$ (11)

When $k = 2$ in equation (8),

$$3u_3 = u_{2tt} + 2\left(\frac{1}{2!}\right) + 2\int_0^t [(t-s)\,u_2(x\,,s)]ds$$

That is; $u_3 = \frac{1}{6}\cos t$ (12)

When $k = 3$ in equation (8)

$$4u_4 = -\frac{1}{6}\cos t + 2\left(\frac{1}{3!}\right) - 2\int_0^t \left[(t-s)\left(\frac{1}{6}\cos[s]\right)\right] ds$$

That is, $u_4 = \frac{1}{24}\cos t$ (13)

Hence;$u(x\,,t) = \sum_{n=0}^{\infty} u = u_0 + xu_1 + x^2u_2 + x^3u_3 + x^4u_4 + \ldots\ldots x^n u_n$

Therefore;

$$u(x\,,t) = \cos t\left[\,1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\ldots\ldots\ldots\right] \qquad (14)$$

Example 2

Consider the equation;

$$xu_x = u_{tt} + x\sin t\int_0^t \sin(t-s)u\,(x\,,s)ds \qquad (15)$$

With Boundary Condition $u(1,t) = t$

Solving equation (15) by (RTDM) is as follows;

$$xu_x = u_{tt} + x\sin t + \int_0^t \sin(t-s)\,u(x,s)ds$$

With Boundary Condition $u(1,t\,) = t$

$$[\,\partial(k-1)](k+1)u_{k+1} = u_k tt + [\,\partial(k-1)]\,\sin t + \int_0^t \sin(t-s)\,u_k(x\,,s)ds \qquad (16)$$

From Boundary Condition, $u_1 = t$, we have $u_{1t} = 1$ and $u_{1tt} = 0$ (17)

When $k = 1$ in equation (16)

$$2u_2 = u_k tt + \sin t + \int_0^t \sin(t-s)\,u_1(x\,,s)ds$$

That is;$u_2 = -\frac{t}{2}$ (18)

We have for $k = 2, 3, \ldots, \infty$, that $u_3 = u_4 = u_\infty = 0$

Hence; $u\,(x\,,t) = \sum_{n=0}^{\infty} u = u_0 + xu_1 + x^2u_2 \ \ldots\ldots\ldots$

Therefore;

$$u\,(x\,,t) = xt - x^2\frac{t}{2} + \ldots\ldots\ldots\ldots \qquad (19)$$

Example 3

Consider the equation;

$$u_t - u_{xx} + u\int_0^t e^{t-s}\,u(x\,,s)ds = (x^2+1)e^t - 2 \qquad (20)$$

With Boundary Condition $u(0,t\,) = t\,;u_x(0,t) = 0$

Solving equation (20) by (RTDM) is as follows;

$$u_t - u_{xx} + u + \int_0^t e^{(t-s)}\,u(x\,,s)ds = (x^2+1)e^t - 2$$

$(k+1)(k+2)u_{k+2}$

$= u_{kt} + u_k - [\delta(k-2)+1]e^t + 2 + \int_0^t e^{(t-s)}\, u_k(x\,,s)ds$ (21)

From the boundary condition we have;

$u_0 = u(0,t) = t;\ u_1 = u_x(0,t) = 0$ (22)

When $k = 0$ in equation (21)

$2u_2 = u_{0t} + u_0 - [\delta(k-2)+1]e^t + 2 + \int_0^t e^{(t-s)}\, u_0(x\,,s)ds$

That is;$u_2 = 1$ (23)

When $k = 1$ in equation (21)

$6u_3 = u_{1t} + u_1 - [\delta(k-2)+1]e^t + 2 + \int_0^t e^{(t-s)}\, u_1(x,s)ds$

That is;$u_3 = \frac{1}{3} - \frac{e^t}{6}$ (24)

When $k = 2$ in equation (21)

$12u_4 = u_{2t} + u_2 - [\delta(k-2)+1]e^t + 2 + \int_0^t e^{(t-s)}\, u_2(x,s)ds$

That is;$u_4 = \frac{1}{6} - \frac{e^t}{12}$ (25)

Hence; $u(x\,,t) = \sum_{n=0}^{\infty} u = u_0 + xu_1 + x^2u_2 + x^3u_3 + x^2u_4 + \ldots\ldots\ldots$

Therefore;

$u(x,t) = t + x^2 + \left[\frac{1}{3} - \frac{e^t}{6}\right]x^3 + \left[\frac{1}{6} - \frac{e^t}{12}\right]x^4 + \ldots\ldots\ldots$ (26)

4.0 RESULTS

The computational results obtained for solving the Partial Integro Differential equation (PIDE) using Reduced Differential Transform Method (RDTM), in the three examples (for$x = 0.05$) for some values of $t = 0.1, 0.2, 0.3, \ldots, 1.0$is shown in the tables below;

Table 4.1: The comparison of the Exact solution and RDTM at when $x = 0.05$ for each value of$t = 0.1, 0.2, 0.3, \ldots, 1.0$. Forexample1 above

t	**Exact** $u = e^x \cos t$	**RDTM**	**Absolute Error**
0, 1	1.05126949520	1.0512694957	$2.63e^{-9}$
0, 2	1.05126469167	1.05126468905	$2.612e^{-9}$
0, 3	1.05124668581	1.05125668319	$2.62e^{-9}$
0, 4	1.05124547764	1.05124547502	$2.62e^{-9}$
0, 5	1.05123106719	1.05123106457	$2.62e^{-9}$
0, 6	*1.05121345451*	*1.05121345189*	$2.62e^{-9}$
0, 7	*1.05119263966*	*1.05119263703*	$2.63e^{-9}$
0, 8	*1.05116862268*	*1.05116862006*	$2.62e^{-9}$
0, 9	*1.05114140367*	*1.05114140104*	$2.63e^{-9}$
1. 0	*1.05111098270*	*1.05111098007*	$2.63e^{-9}$

Table 4.2: The comparison of the Exact solution and RDTM at when $x = 0.05$ for each value of $t = 0.1, 0.2, 0.3, \ldots, 1.0$. For example 2 above

t	**Exact** $u = xt$	**RDTM**	**Absolute Error**
0, 1	*0.00500000*	*0.00487500*	$1.25e^{-4}$
0, 2	*0.01000000*	*0.00975000*	$2.5e^{-4}$
0, 3	*0.01500000*	*0.01462500*	$3.75e^{-4}$
0, 4	*0.02000000*	*0.01950000*	$5.0e^{-4}$
0, 5	*0.02500000*	*0.02437500*	$6.25e^{-4}$
0, 6	*0.03000000*	*0.02925000*	$7.5e^{-4}$
0, 7	*0.03500000*	*0.03412500*	$8.75e^{-4}$
0, 8	*0.04000000*	*0.03900000*	$1.0e^{-3}$
0, 9	*0.04500000*	*0.04387500*	$1.125e^{-3}$
1. 0	*0.05000000*	*0.04875000*	$1.25e^{-3}$

Table 4.3: The comparison of the Exact solution and RDTM at when $x = 0.05$ for each value of $t = 0.1, 0.2, 0.3, \ldots, 1.0$. For example 3 above

t	**Exact** $u = x^2 + t$	**RDTM**	**Absolute Error**
0, 1	*0.10250000*	*0.10251910833*	$1.910833e^{-5}$
0, 2	*0.20250000*	*0.2025166263*	$1.66263e^{-5}$
0, 3	*0.30250000*	*0.30251388322*	$1.388322e^{-5}$
0, 4	*0.40250000*	*0.40251085166*	$1.085166e^{-5}$
0, 5	*0.50250000*	*0.50250750126*	$7.50126e^{-6}$
0, 6	*0.60250000*	*0.6025037985*	$3.7985e^{-6}$
0, 7	*0.70250000*	*0.70249970632*	$2.9368e^{-7}$
0, 8	*0.80250000*	*0.80249518376*	$4.81624e^{-6}$
0, 9	*0.90250000*	*0.90249018556*	$9.81444e^{-6}$
1.0	*1.00250000*	*1.00248466169*	$1.533831e^{-5}$

5.0 CONCLUSION

In this research work the Reduced Differential Transform Method (RDTM) is successfully used to develop approximate solution to Partial Integro Differential equation. Approximation can be obtained to any desired number of terms to increase the level of accuracy.

The reliability and consistency of the method gives it a wide range of applicability. The results show that this method, RDTM is easy and serves as an alternative solving method to the existing techniques of solving non-linear partial differential equations.

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