

STABILITY OF ONE-STEP SCHEME INTERPOLATION FUNCTION IN ORDINARY DIFFERENTIAL EQUATIONS.

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Abstract

In this research work, we consider the stability of one-step scheme interpolation function in ordinary differential equations of the form:

$F(x) = a_0 + a_1x + b \tan(\rho x + \sigma)$, where a_0 , a_1 and b are real undetermined coefficient, ρ and σ are complex parameters. We consider the linear multistep method in the

$\sum_{j=0}^k \alpha_j y_{n+j} = h \sum_{j=0}^k \beta_j f_{n+j}$ where α_j and β_j are constants and $\alpha_k \neq 0$. The Taylor's

series expansion of $y(x_{t+1})$ about $x = x_t$ is $y(x_{t+1}) = y(x_t) + \sum_{i=1}^{\infty} \frac{h^i y^{(i)}(x_t)}{i!}$ was also

implemented to establish our result at $k=5$.

Keyword: Interpolation Function, Stability, Consistency and Convergence

1.0 INTRODUCTION

In this research work, we shall consider one-step scheme interpolation function for solving Initial Value Problem (IVP) in ordinary differential equation in the form:

$$y' = f(x, y), y(x_0) = y_0, \quad a \leq x \leq b \tag{1}$$

The development of numerical methods for singular IVP systems of ordinary differential equations (ODEs) have been attracting much attention due to their needs in the solutions of problems arising from the mathematical formulation of physical situations in singular perturbation, chemical kinetics population models, mechanical oscillation, process control and electrical circuit theory which often leads in initial value problems (IVPs) in ordinary differential equation that are singular.

Many researchers have done a great deal of work in this area. Worthy of note are those of [1-13]. In their different approaches, they show the method of one-step technique of interpolation function for solving singular initial value problem (ivp) in ordinary differential equation.

The problem for this research work is to find a numerical solution to the (IVP) which is represented by $y' = f(x, y)$, $y(x_0) = y_0$, $a \leq x \leq b$ where $f(x, y)$ is defined and continuous in a region $D \subset (a, b)$ that are singular and $f(x, y)$ also satisfy a Lipschitz condition with respect to y .

However, our aim in this research work is to implement the problem represented by (1), where $f(x, y)$ must satisfy a Lipschitz condition with respect to y . To achieve our aims, we therefore set the following objectives;

- (i) To develop an interpolation function of singular ivp in the form.
 $F(x) = a_0 + a_1x + b \tan(\rho x + \sigma)$.
- (ii) To determine the performance, stability, characteristics, consistency and nature of convergence constructed in (i) above.
- (iii) To develop a Fortran package for the implementation of our scheme.

2.0 SOME DEFINITIONS AND NOTATIONS

For the purpose of this research work, we have the following definitions.

(1) Definition of Consistency [9]

$$\text{The general one-step method } F(x) = a_0 + a_1 x + b \tan(\rho x + \sigma) \tag{2}$$

is said to be consistent with the initial value problem in equation (1) if $\phi(x, y, 0) = f(x, y)$. If the method in equation (1) is consistent with the initial-value problem, then $y(x+h) - y(x) - h\phi(x, y(x), 0) + O(h^2) = 0(h^2)$. Since $y'(x) = f(x, y(x)) = \phi(x, y(x), 0)$, by definition (1). Thus a consistent method has order of at least one. The only linear multi-step method which falls within the class in equation (1) is Euler's rule which is obtained by setting $\phi(x, y, h) = \phi_E(x, y, h) = f(x, y)$ (The subscript E denotes "Euler"). The consistency condition of definition (1) is then obviously satisfied and a simple calculation shows that the order according to definition (1) is one.

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(2) Definition of Local Truncation Error [10]

The local truncation error at x_{n+1} of the general explicit one-step method in equation (2) above is defined to be T_{n+1} where $T_{n+1} = y(x_{n+1}) - h\phi(x_n, y(x_n), h)$ and (x) is the theoretical solution of the initial value problem. Then the local truncation error at x_{n+k} of an explicit linear K -step method satisfies $T_{n+k} = y(x_{n+k}) - y_{n+k}$. The relationship between global and local truncation error is $e_{n+1} \leq K T_{n+1}$ where K is a constant. The local truncation error is directly proportional to the global error introduced at each step mostly when the derivation and computation of local truncation error is rigorous and all previous solutions are exact.

(3) Definition of Convergence [9]

The general one-step method equation (2) above is said to be convergent to the initial value problem in equation (1) if the corresponding approximation y_n satisfies $y_n \rightarrow y(x_n)$ as $n \rightarrow \infty$

(4) The function $f(x, y)$ in (1) is said to satisfy a **Lipschitz condition** in y , over a region D , if there exists a constant L such that

$$\|f(x, y_1) - f(x, y_2)\| \leq L \|y_1 - y_2\| \quad (3)$$

In this case, L is called the Lipschitz constant and $f(x, y)$ is said to be Lipschitzian.

(5) A point x_n in the interval (a, b) at which y_n is being computed is called a **mesh point**.

(6) The difference, $h = x_{n+1} - x_n$ between the current point x_n and the next point x_{n+1} is called the **mesh size** of the subinterval

$$(x_n, x_{n+1}) \subset (a, b).$$

(7) Region of absolute stability (RAS):

A region D of the complex plane is said to be a region of absolute stability of a given method, if the method is absolutely stable for $\bar{h} \in D$ for a one-step integrator, D is RAS if $|s(\bar{h})| < 1$ for every $\bar{h} \in D$.

(8) Interval of absolute stability (IAS):

An interval (α, B) of the real line is said to be an interval of absolute stability of the method if the method is absolutely stable for all $\bar{h} \in (\alpha, B)$. If the method is obviously unstable for all \bar{h} it is said to have no interval of absolute stability (IAS). The interval of absolute stability is determined by the coefficients of the method.

(9) A-STABLE

A one-step method is said to be **A-STABLE** if when applied to the test equation $y' = \lambda y$ with

$$\text{Re}(\lambda) < 0, \text{ it gives, } y_{n+1} = S(\bar{h})y_n, \text{ with the stability function } S(\bar{h}) \text{ satisfying, } |S(\bar{h})| < 1 \text{ for all } \text{Re}(\bar{h}) < 0, \bar{h} = \lambda h$$

(10) L-Stable

A given One-Step Method is said to be **L-Stable** if it is A-Stable and in addition,

$$\text{Re}(\bar{h}) \xrightarrow{\text{Limit}} -\infty |S(\bar{h})| = 0$$

(11) A(α) - STABLE

A numerical integrator method is said to be **A(α) - STABLE** for some $\alpha \in [0, \pi/2]$ if the infinite wedge. $S_\alpha = \{Z: |\text{Arg}(-Z)| < \alpha, Z \neq 0\}$ i.e. $Z = \bar{h}$ is contained in the region of absolute stability.

(12) A(0) - Stable

A numerical method is said to be **A(0) - Stable** if it is A(α) - Stable for some $\alpha \in [0, \pi/2]$. $A(\pi/2)$ - Stability \equiv A - stability.

3.0 THE DERIVATION

In this research work, we shall develop an interpolation function of singular initial value problem in (1) by using a function.

$$F(x) = a_0 + a_1 x + b \tan(\rho x + \sigma) \quad (4)$$

where a_0, a_1 and b are real undetermined coefficient, ρ and σ are complex parameters.

$$\text{We define } \theta_x = \rho x + \sigma \quad (5)$$

Substitute (5) in (4), we have

$$F(x) = a_0 + a_1 x + b \tan \theta(x) = y \quad (6)$$

$$F(x_t) = a_0 + a_1 x_t + b \tan \theta_t = y_t \quad (7)$$

$$F(x_{t+1}) = a_0 + a_1 x_{t+1} + b \tan \theta_{t+1} = y_{t+1} \quad (8)$$

Set $\theta_{t+1} = \theta_t + \rho h$

Therefore (8) becomes

$$F(x_{t+1}) = a_0 + a_1 x_{t+1} + b \tan(\theta_t + \rho h) \quad (9)$$

Differentiating (7), we have

$$F'(x_t) = a_1 + b (\sec^2 \theta_t) \rho = f_t \quad (10)$$

$$= a_1 + \rho h (1 + \tan^2 \theta_t) = f_t \tag{11}$$

Differentiating (10) we have,

$$F'_{(x_t)} = 2\rho^2 b \tan \theta_t \sec^2 \theta_t = f_t^{(1)} \tag{12}$$

Making b the subject of the formula we have,

$$b = \frac{f_t'}{2\rho^2 \tan \theta_t \sec^2 \theta_t} \tag{13}$$

$$\text{From (11) } a_1 = f_t - \rho h (1 + \tan^2 \theta_t) \tag{14}$$

Putting (13) into (14), we have

$$a_1 = f_t - \frac{f_t' \rho (1 + \tan^2 \theta)}{2\rho^2 \tan \theta_t \sec^2 \theta_t}$$

$$a_1 = f_t - \frac{f_t^{(1)}}{2\rho^2 \tan \theta_t} \tag{15}$$

Substituting (8) from (7), we have

$$F_{(x_{t+1})} - F_{(x_t)} = a_1 h + b(\tan \theta_{t+1} - \tan \theta_t) \tag{16}$$

Putting (15) into (16), we have

$$F_{(x_{t+1})} - F_{(x_t)} = h \left\{ f_t - \frac{f_t^{(1)}}{2\rho \tan \theta_t} \right\} + b \{ \tan \theta_t + \tan(\rho h) - \tan \theta_t \}$$

$$F_{(x_{t+1})} - F_{(x_t)} = h \left\{ f_t - \frac{f_t^{(1)}}{2\rho \tan \theta_t} \right\} + b \left\{ \frac{\tan \rho h + \tan^2 \theta \tan(\rho h)}{1 - \tan \theta_t + \tan(\rho h)} - \tan \theta_t \right\}$$

$$F_{(x_{t+1})} - F_{(x_t)} = h \left\{ f_t - \frac{f_t^{(1)}}{2\rho \tan \theta_t} \right\} + b \left\{ \frac{\tan \theta_t + \tan \rho h - \tan \theta_t + \tan^2 \theta (\rho h)}{1 - \tan \theta_t + \tan(\rho h)} \right\}$$

$$F_{(x_{t+1})} - F_{(x_t)} = h \left\{ f_t - \frac{f_t^{(1)}}{2\rho \tan \theta_t} \right\} + b \left\{ \frac{\tan \rho h + \tan^2 \theta \tan \rho h}{1 - \tan \theta_t + \tan(\rho h)} \right\} \tag{17}$$

Substituting $b = \frac{f_t^{(1)}}{2\rho^2 \tan \theta_t \sec^2 \theta_t}$ in the above expression, we have

$$F_{(x_{t+1})} - F_{(x_t)} = h \left\{ f_t - \frac{f_t^{(1)}}{2\rho \tan \theta_t} \right\} + \frac{f_t^{(1)}}{2\rho^2 \tan \theta_t \sec^2 \theta_t} \frac{(1 + \tan^2 \theta_t) \tan(\rho h)}{(1 + \tan \theta_t \tan(\rho h))}$$

$$F_{(x_{t+1})} - F_{(x_t)} = h \left\{ f_t - \frac{f_t^{(1)}}{2\rho \tan \theta_t} \right\} + \frac{f_t^{(1)} \tan(h)}{2\rho^2 \tan \theta_t + (1 - \tan \theta_t \tan \rho h)}$$

$$\therefore y_{t+1} - y_t = h \left\{ f_t - \frac{f_t^{(1)}}{2\rho \tan \theta_t} \right\} + \frac{f_t^{(1)} \tan(h)}{2\rho^2 \tan \theta_t + (1 - \tan \theta_t \tan \rho h)} \tag{18}$$

ρ and θ_t are determined by using the definition of local truncated error denoted by

$$T_{t+1} = y(x_{t+1}) - y_{t+1} \tag{19}$$

Where $y(x_{t+1})$ is the theoretical solution at $x = x_{t+1}$ and y_{t+1} is the numerical solution obtained by adopting (18). Taylor's series expansion of $y(x_{t+1})$ about $x = x_t$ is

$$y(x_{t+1}) = y(x_t) + \sum_{i=1}^{\infty} \frac{h^i y^{(i)}(x_t)}{i!} \tag{20}$$

With the assumption that there was no previous error $y_t = y(x_t)$ as well as using Macularin series expansion of $\tan(\rho h)$ and binomial expansion of $(1 - \tan \theta_t \tan^3 \rho h)^{-1}$. we have

$$(1 - \tan \theta_t \tan \rho h)^{-1} = 1 + \tan \theta_t \tan(\rho h) + \tan^2 \theta_t \tan^2(\rho h) + \tan^3 \theta_t \tan^3(\rho h)$$

$$\tan(\rho h) = h + \frac{(\rho h)^3}{3} + \frac{2(\rho h)^5}{15} + 17 \frac{(\rho h)^7}{315} +$$

$$\tan^2(\rho h) = (\rho h)^2 + \frac{2}{3}(\rho h)^4 + \frac{17(\rho h)^6}{45} + \frac{2790(\rho h)^8}{14175} + \frac{4810(\rho h)^{10}}{7875} \tag{21}$$

$$\tan^3(\rho h) = (\rho h)^3 + (\rho h)^5 + \frac{11}{15}(\rho h)^7 + \dots$$

$$(1 - \tan \theta_t \tan(\rho h))^{-1} = 1 + (\rho \tan \theta_t)h + (\rho^2 \tan^2 \theta_t)h^2 + \left(\frac{\rho^3}{3} \tan \theta_t + \rho^3 \tan \theta_t\right)$$

$$h^3 + \left(\frac{2}{3}\rho^4 \tan \theta_t\right)h^4 + \left(\frac{2}{15}\rho^5 \tan \theta_t + \rho^5 \tan^2 \theta_t\right)h^5 + \frac{17}{45}\rho^6 \tan^2 \theta_t + \dots \tag{22}$$

$$\therefore y_{t+1} - y_t = h \left\{ f_t - \frac{f_t^{(1)}}{2\rho \tan \theta_t} \right\} + f_t^{(1)} \{ \rho h + \rho^2 \tan \theta_t \} h^2$$

$$\begin{aligned}
 & + \left(\frac{\rho^3}{3} + \rho^3 \tan \theta_t \right) h^3 + \left(\frac{2}{3} \rho^4 \tan \theta_t + \rho^4 \tan^3 \theta \right) h^4 \\
 & + \left(\frac{2}{15} \rho^5 + \rho^5 \tan^2 \theta \right) h^5 + \frac{\left(\frac{17}{45} \rho^6 \tan \theta_t + \frac{4}{3} \rho^6 \right)}{2 \rho^2 \tan \theta_t} h^6
 \end{aligned} \tag{23}$$

$$y_{t+1} = y_t + h y_t^{(1)} + \frac{h^2 y_t^{(2)}}{2!} + \frac{h^3 y_t^{(3)}}{3!} + \dots \tag{24}$$

Equating coefficient of h's in (23) and (24) we have,

$$\begin{aligned}
 f_t^{(1)} \frac{\left(\frac{1}{3} \rho^3 + \rho^3 + \tan \theta_t \right)}{2 \rho^2 \tan \theta_t} &= \frac{f_t^{(2)}}{3!} \\
 f_t^{(2)} \rho \left(\frac{1}{3!} + \tan^2 \theta_t \right) &= \frac{f_t^{(2)} \tan \theta_t}{3}
 \end{aligned} \tag{25}$$

$$\begin{aligned}
 f_t^{(1)} \frac{\left(\frac{2}{3} \rho^4 \tan \theta_t + \rho^4 + \tan^3 \theta_t \right)}{2 \rho^2 \tan \theta_t} &= \frac{f_t^{(3)}}{4!} \\
 f_t^{(1)} \rho^2 \left(\frac{2}{3} \tan^2 \theta_t \right) &= \frac{f_t^{(3)}}{12}
 \end{aligned} \tag{26}$$

$$\begin{aligned}
 T_{n+1} &= h^5 \frac{\left(\frac{2}{15} \rho^5 + \rho^5 + \tan^2 \theta_t \right)}{2 \rho^2 \tan \theta_t} f_t^{(1)} - \frac{f_t^{(4)}}{5!} h^5 \\
 &= h^5 \rho^3 \frac{\left(\frac{2}{15} + \tan^2 \theta_t \right)}{2 \tan \theta_t} f_t^{(1)} - \frac{f_t^{(4)}}{120} h^5
 \end{aligned}$$

4.0 THE COMPUTATIONAL ANALYSIS OF THE STABILITY OF ONE-STEP SCHEME INTERPOLATION FUNCTION

Problem 1: Our method, consider $y^1 = 1 + y^2$, with mesh size $h = 0.05$

Table 1: Experimental Result of our scheme

X	H	Theoretical solution Y_t	Numerical Solution Y_{t+1}	Error
0.05	0.05	1.10535551	1.1054390	.763367369x10 ⁻⁴
0.10	0.05	1.22304879	1.2232060	.128508053x10 ⁻³
0.15	0.05	1.35608774	1.3563164	.168551889x10 ⁻³
0.20	0.05	1.50849752	1.5087907	.200243872x10 ⁻³
0.25	0.05	1.68579627	1.6861813	.228325494x10 ⁻³
0.30	0.05	1.89576495	1.8962732	.268042910 x10 ⁻³
0.35	0.05	2.14974943	2.1504447	.324247923 x10 ⁻³
0.40	0.05	2.46496249	2.4659612	.405001097 x10 ⁻³
0.45	0.05	2.86888368	2.8708886	.524302857 x10 ⁻³
0.50	0.05	3.40822297	3.4106376	.707985416 x10 ⁻³
0.55	0.05	4.16936335	4.1735691	.100771160 x10 ⁻²
0.60	0.05	5.33185411	5.3400701	.153856182 x10 ⁻²
0.65	0.05	7.34043450	7.3595575	.259838399 x10 ⁻²
0.70	0.05	11.6813686	11.741792	.514599444 x10 ⁻¹
0.75	0.05	28.2382227	28.592778	.124001812 x10 ⁻¹
0.80	0.05	68.4798454	66.498083	.298017906 x10 ⁻¹

Table 2: Lambert and Shaw [10], h = 0.1

X	Theoretical solution Y	- A (n)	N (n)	Initial Solution Error	Improved solution
0.1	1.223048,880	0.871,0524	-1.459,5387	2x10 ⁻⁷	2x10 ⁻⁸
0.2	1.508,497,647	0.818,6067	-1.2095811	5x10 ⁻⁷	3x10 ⁻⁸
0.3	1.895,765,123	0.797,0428	-1.03901411	1x10 ⁻⁶	6x10 ⁻⁸
0.4	2.464,962,757	0.7887937	-1.032,812,1	3x10 ⁻⁶	1x10 ⁻⁷
0.5	3.408,223,442	0.7861141	-1.009,3671	5x10 ⁻⁶	2x10 ⁻⁷
0.6	5.331,855,223	0.7854784	-1.001, 6126	1x10 ⁻⁵	5x10 ⁻⁵
0.65	7.354,436,575	0.7854150	-1.000,4736	3x10 ⁻⁵	1x10 ⁻⁶
0.70	11.681,373,800	0.785,4002	-1.0000712	7x10 ⁻⁵	3x10 ⁻⁶
0.75	28.238,232,850	0.785,3987	-1.000020	4x10 ⁻⁴	4x10 ⁻⁵

Table 3: Shaw [11], $h = 0.05$

X	Theoretical solution Y	Error in numerical solutions	
		Rational (k=1)	Rational (k=2)
0.1	1.223048,888	8×10^{-8}	1×10^{-4}
0.2	1.508,497,647	2×10^{-7}	3×10^{-4}
0.3	1.895,765,123	4×10^{-7}	6×10^{-4}
0.4	2.464,962,757	7×10^{-7}	1×10^{-4}
0.5	3.408,223,442	1×10^{-6}	3×10^{-3}
0.6	5.331,855,223	4×10^{-6}	7×10^{-2}
0.65	7.354,436,575	8×10^{-6}	1×10^{-2}
0.70	11.681,373,800	2×10^{-5}	4×10^{-2}
0.75	28.238,252,830	1×10^{-4}	3×10^{-1}

Problem 2: Consider $y' = y, y(0) = 1, 0 \leq x \leq 1$ with exact solution $y = \exp(x), h = 0.1$

Table 4

Xn	H	Y _t	Y _{t+1}	e _n
0.1	0.1	1.105170965	1.105170919	0.000000047
0.2	0.1	1.221402764	1.210341837	0.011060930
0.3	0.1	1.349858880	1.315512763	0.034346110
0.4	0.1	1.491824746	1.420683674	0.071141070
0.5	0.1	1.648721218	1.525854586	0.122866600
0.6	0.1	1.822118878	1.631025527	0.191093400
0.7	0.1	2.013752699	1.736196409	0.277556300
0.8	0.1	2.225541115	1.841367348	0.384173800
0.9	0.1	2.459603310	1.94 6538288	0.513065000
1.0	0.1	2.718282223	2.051709171	0.666573000

Table 5: Aashikpelokhai [1] $y' = y, y(0) = 1, 0 \leq x \leq 1$ with exact solution $y = \exp(x), h = 0.1$

Xn	Y _t	Y _{t+1}	En	Order 5	Order 4	Order 3
0.1	1.10517092	1.10517	9.2×10^{-7}	-0.15622	0.60450	1.57761
0.2	1.22140028	1.22140	2.8×10^{-7}	-0.17265	0.48289	1.74353
0.3	1.34985881	1.34985	4.7×10^{-6}	-0.19081	0.38781	1.92690
0.4	1.49182470	1.49182	1.2×10^{-6}	-0.21085	0.36697	2.12955
0.5	1.64872127	1.64872	8.8×10^{-6}	-0.23305	0.321056	2.35355
0.6	1.82211880	1.82211	2.7×10^{-6}	-0.25756	0.02256	2.60104
0.7	2.01375271	2.01375	9.3×10^{-7}	-0.28465	0.35937	2.87460
0.8	2.22554093	2.22554	3.1×10^{-6}	-0.31459	0.34473	3.17692
0.9	2.45960311	2.45960	1.8×10^{-6}	-0.34765	0.30697	3.51104
1.0	2.71828183	2.71828	1.8×10^{-6}	-0.38424	0.03830	3.88030

5.0 CONCLUSION

In conclusion, from the result of the analysis of the stability of one-step scheme interpolation function method in tables 1,2,3,4 and 5 above, we discover it is A-stable . For A-stable the Region of the Absolute Stability (RAS) together with the point of encroachment into the left – half of the complex plane is stable.

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