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MATHEMATICAL MODEL OF BACTERIA-NUTRIENT HARVESTING IN A CULTURED ENVIRONMENT

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Abstract

In this paper, we develop mathematical model of bacteria-nutrient harvesting in a cultured environment. This model which assumes that the rate of harvesting of these bacteria is constant results in a system of first order differential equations. Analyzing the model, it was discovered that the product of the maximum nutrient uptake per cell and the number of cells produced per unit of nutrient uptake is constant $(VY = ln2 + h)$.
It was also assumed that the rate of harvesting of these bacteria varies and a corresponding model was developed. Analyzing this model using methods from dynamical systems theory, it was seen that the system has two steady states. The first steady state is unstable while the second is globally asymptotically stable if the carrying capacity of the environment has a lower bound, which is a ratio of the harvesting coefficient of the bacteria, cost per unit effort per unit price of the bacteria.

Keywords: Bacteria, Dynamical Systems, Steady State, Cultured Environment, Variable Effort

1. INTRODUCTION

The economic importance of bacteria motivates us to investigate the behavioural pattern of bacteria growth in response to harvesting in a cultured environment. In this investigation, we assume that bacteria in a culture can grow to extinction as a result of some conditions unfavourable to their growth. One of these unfavourable conditions is the exertion of some acidic substance as a result of cell's growth. We also assume that the bacteria which have logistic growth are cultured for industrial use and should be harvested from time [1].

Bacteria can grow in a spatial pattern in response to diffusion of needed nutrient. This describes an immobile bacteria population distribution on a petri-dish. The diffusing nutrient is taken up by growing cell and acid is produced as a product of cell's growth [1,2].

Some of the major works have been done on mathematical modeling of bacteria nutrient dynamics.

In [3], the dynamics of the bacteria nutrient and Chemoattractant is modeled; this gives rise to a system of non-linear partial differential equations. In this study, it was observed that pattern arise from disturbances to a spatially uniform solution state. A linear analysis gave rise to a second order ordinary differential equation for the amplitude of each mode present in the initial disturbance.

In [4], a new chemostat model with continuous microbial culture and harvest, which investigates the dynamics of the system, was presented. Different from the conventional ones, this model includes a constant periodic flocculant transmission. By using theory of impulsive differential equations, it was shown that the microbe-extinction periodic solution is globally asymptotically stable when a threshold value is less than 1, and system is permanent when a certain threshold value is greater than 1. Then, according to the threshold associated with microbial extinction or existence, the control strategy for microbial continuous cultivation and harvest was discussed. Under such control strategy, continuous microbial culture and harvest can be achieved by adjusting input time, input amount or concentration of the flocculant. Finally, an example with numerical simulations was given to illustrate the theoretical conclusions.

In [5], a simple system of delay differential equations (DDEs) for quorum sensing of Pseudomonas putida with one positive feedback plus one (delayed) negative feedback mechanism is presented.

Results were shown concerning fundamental properties of solutions, such as existence, uniqueness, and non-negativity; the last feature is crucial for mathematical models in biology and is often violated when working with DDEs. The qualitative behavior of solutions was investigated, especially the stationary states and their stability.

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It was shown that for a certain choice of parameter values, the system presents stability switches with respect to the delay. On the other hand, when the delay is set to zero, a Hopf bifurcation might occur with respect to one of the negative feedback parameters. Model parameters were fitted to experimental data, indicating that the delay system is sufficient toexplain and predict the biological observations

In [6], the mathematical model of bacteria nutrient dynamics which resulted in a system of first order ordinary differential equations is developed. The analysis of the model was done using dynamical systems theory. It was found out that the product of maximum nutrient uptake by a growing cell is a constant($VY = ln2$). It was also shown that there is a linear relationship between the concentration of limiting nutrient and the number of bacteria with a negative slope.

In [7], the mathematical model of fish harvesting in a common access fishery is presented. According to this work the fisherman continues to make harvest as long as his net income is positive. It was assumed that the fish has a logistic growth. The model was formulated and analyzed using methods from dynamical systems theory. The analysis showed that model has two steady states; the first is unstable while the second is globally asymptotically stable.

2. MODEL FORMULATION

In this paper we present a mathematical model of bacteria-nutrient harvesting in a culture which the farmer continues to harvest as long as his net income is positive. Here we assume that the rate of harvesting of bacteria is constant (a fixed quantity of bacteria is harvested at a given period of time).

The model in [3] is modified as follows:

$$\frac{dB}{dt} = YBN_u - hB \qquad (2.1)$$

$$\frac{dN}{dt} = -BN_u \qquad (2.2)$$

$B(t) = Number\ of\ bacteria\ attimet$

$N(t) = Concentration\ of\ limiting\ nutrient$

$Y = The\ yield, that\ is\ the\ number\ of\ cells\ produced\ perunit\ of\ nutrient\ taken\ up$

$t = Time, and\ is\ chosen\ to\ correspond\ to\ cell's\ doubling\ time\ when\ suffient\ nutrient\ is\ taken$

$V = Maximum\ nutrient\ taken\ up\ per\ cell$

$K = Michaeli's\ or\ saturation\ constant\ which\ is\ the\ value\ of\ Nat\ which\ up\ take\ is\ half\ it's\ maximum\ rate$

$h = harvestingrateofbacteria$

Jacob-Monod is model for nutrient uptake is described by

$$N_u = \frac{VN}{K + N}$$

The equation (2.1) and (2.2) become

$$\frac{dB}{dt} = \frac{YBVN}{K + N} - hB \qquad (2.3)$$

$$\frac{dN}{dt} = \frac{BVN}{K + N}$$

3.0 ANALYSIS OF THE MODEL

We assume that nutrient is sufficiently present ($N > 1$), hence we have $\frac{N}{K + N} \approx 1$

Then

$$\frac{dB}{dt} = YBV - hB \qquad (3.1)$$

$$\frac{dB}{B} = (YV - h)dt$$

$$LnB = (YV - h)t + C$$

$$B(t) = e^{(YV-h)t + C}$$

At $t = 0$

$$B(t) = B_0 e^{(YV-h)t}$$

Since the time scale corresponds to cell doubling, then

$$B(1) = B_0 e^{(YV-h)1} = 2B(0)$$

$$\Rightarrow B_0 e^{(YV-h)1} = 2B(0)$$

$$\Rightarrow e^{(YV-h)} = 2$$

$$\Rightarrow YV - h = Ln2$$

$$\Rightarrow VY = Ln2 + h \qquad (3.2)$$

Determining the population of bacteria

$$\frac{dB}{dN} = \frac{-YBVN + khB + NhB}{BVN}$$

$$\frac{dB}{dN} = \frac{-YVN + kh + Nh}{VN}$$

$$\frac{dB}{dN} = -Y + \frac{kh}{VN} + \frac{h}{V}$$

$$dB = -YdN + \frac{kh}{VN}dN + \frac{h}{V}dN$$

Integrating from 0 to t

$$B(t) = \frac{kh}{V}LnN - YN + \frac{h}{V}N + YN_0 - \frac{kh}{V}LnN_0 - \frac{h}{V}N_0 + B_0 \qquad (3.3)$$

4.0 THE MODEL OF VARIABLE EFFORT

We assume that:

1. Effort in harvesting of bacteria is not constant but varies with time t.
2. Effort increases with positive income
3. The bacteria in the cultured environment has a logistic growth
4. Harvesting is influence by the revenue that accrues.

PARAMETERS

$\gamma - percapitagrowthofbacteria$

$E(t) - harvestingeffortattimet$

$R(t) - Revenuefromharvestingattimet$

$c - costperuniteffort$

$\alpha - constantofproportionality$

$\rho - priceperunitquantityofbacteria$

$\delta - harvestingcoefficientofbacteria$

$T - carryingcapacityoftheenvironment$

Using the above assumptions and parameters, we write the model equation as follows:

$$\frac{dB}{dt} = \gamma B\left(1 - \frac{B}{T}\right) - \delta EB \qquad (4.1)$$

$$\frac{dE}{dt} = \alpha R = \alpha E(\delta\rho B - c)$$

Where

$R = E(\delta\rho B - c)$

That is, effort is proportional to the revenue (net income) at time t

STEADY STATES AND STABILITY

The steady state of the model exist at

$$\frac{dB}{dt} = \frac{dE}{dt} = 0$$

$$\frac{dB}{dt} = 0 \implies \gamma B\left(1 - \frac{B}{T}\right) - \delta EB = 0$$

$$\frac{dE}{dt} = 0 \implies \alpha R = \alpha E(\delta\rho B - c = 0$$

$$\gamma B\left(1 - \frac{B}{T}\right) - \delta EB = 0 \quad \dots\dots\dots\dots (4.2)$$

$$\alpha E(\delta\rho B - c = 0 \quad \dots\dots\dots..(4.3)$$

Solving (4.2) and (4.3), obtain the steady state values as,

$$E^0 = 0, \qquad B^0 = \frac{c}{\rho\delta}$$

$$E^0 = \frac{\gamma(T\rho\delta - c)}{T\delta^2\rho}, \qquad B^0 = 0$$

Hence, $(B^0, E^0) = (0, 0) and \left(\frac{c}{\rho\delta}, \frac{\gamma(T\rho\delta - c)}{T\delta^2\rho}\right)$

Linearizing the system, we have

$$f_1(B,E) = \gamma B\left(1 - \frac{B}{T}\right) - \delta EB$$

$$f_2(B,E) = \alpha E(\delta\rho B - c)$$

The Jacobean of the system is given as

$$J = \begin{pmatrix} \gamma - \frac{2\gamma B^0}{T} - \delta E^0 & -\delta B^0 \\ \alpha\delta\rho E^0 & \alpha E(\delta\rho B^0 - c) \end{pmatrix}$$

Considering the Jacobian at the first steady state $(B^0\ ,\ E^0) = (0,\ 0)$

$$J_1 = \begin{pmatrix} \gamma & 0 \\ 0 & -\alpha c \end{pmatrix}$$

The eigenvalues of the system at $(B^0\ ,\ E^0) = (0,\ 0)$ are γ and $-\alpha c$

Since $> 0, \alpha > 0,\ and c > 0$, the steady state $(B^0\ ,\ E^0) = (0,\ 0)$ is not stable

Considering the Jacobian at the second steady state $(B^0\ ,\ E^0) = \left(\frac{c}{\rho\delta}\ ,\ \frac{\gamma(T\rho\delta - c)}{T\delta^2\rho}\right)$

$$J_2 = \begin{pmatrix} -\frac{\gamma c}{T\rho\delta} & \frac{-c}{\rho} \\ \frac{\alpha\gamma(T\rho\delta - c)}{T} & 0 \end{pmatrix}$$

The characteristic equation of J_2 is $\lambda^2 + \frac{\gamma c}{T\rho\delta}\lambda + \frac{\alpha\gamma c\,(T\rho\delta - c)}{T\rho} = 0$ (4.4)

$$\lambda = \frac{-\frac{\gamma c}{T\rho\delta} \pm \sqrt{(\frac{\gamma c}{T\rho\delta})^2 - \frac{4\alpha\gamma c(T\rho\delta - c)}{T\rho}}}{2}$$

For λ to exist, then

$$(\frac{\gamma c}{T\rho\delta})^2 - \frac{4\alpha\gamma c(T\rho\delta - c)}{T\rho} \geq 0$$

$$\frac{\gamma c}{T\rho\delta} \geq \sqrt{\frac{4\alpha\gamma c(T\rho\delta - c)}{T\rho}}$$

$$\Rightarrow T\rho\delta - c \geq 0$$

$$\Rightarrow T \geq \frac{1}{\delta}\left(\frac{c}{\rho}\right)$$

5.0 SUMMARY AND CONCLUSION

In this work, we have been able to formulate the mathematical model of bacteria-nutrient harvesting in a cultured environment. This could be an environment where bacteria is cultivated and nurtured for industrial purposes. The model is formulated under certain assumptions.

We first assumed that the harvesting rat is constant, that is, a fixed quantity of bacteria is harvested at a given period of time.

Analyzing the model with this assumption, we discovered that the product of maximum nutrient uptake per cell and the number of cells produced per cell of nutrient uptake is given as

$VY = ln2 + h.$

And the population of bacteria at time t is given as $B(t) = \frac{kh}{V}LnN - YN + \frac{h}{V}N + YN_0 - \frac{kh}{V}LnN_0 - \frac{h}{V}N_0 + B_0$

Secondly, we assume that the rate of harvesting is not constant but depends on the effort exerted in harvesting. This effort is directly proportional to the net income of the bacteria harvesting. This result in a system of ordinary differential equations, analyzing the system using methods from dynamical system theory we obtain the steady state of the system to be; $(B^0$, $E0 = 0,\ 0\ and c\rho\delta\ ,\ \gamma T\rho\delta - cT\delta 2\rho$

At the steady state $(B^0\ ,\ E^0) = (0,\ 0)$, the system is unstable since $\gamma > 0, \alpha > 0,\ and c > 0$

At the steady state $(B^0\ ,\ E^0) = \left(\frac{c}{\rho\delta}\ ,\ \frac{\gamma(T\rho\delta - c)}{T\delta^2\rho}\right)$, the system is globally asymptotically stable if $T \geq \frac{1}{\delta}\left(\frac{c}{\rho}\right)$ where the net income is zero. This means that the steady state is globally asymptotically stable if the carrying capacity of the environment has a lower bound, which is a ratio of the harvesting coefficient of the bacteria, cost per unit effort per unit price of the bacteria.

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