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AN ORDER SEVEN BERNSTEIN INDUCED HYBRID TWO-STEP METHOD FOR DIRECT SOLUTION OF SECOND-ORDER INITIAL PROBLEMS

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Abstract

In this article, a two step Bernstein Hybrid Multistep Method (BHMM) of order seven is developed for the direct solution of second order initial problems. To derive this method, the approximate function was interpolated and collocated at equidistant grid and offgrid points. The continuous scheme was evaluated at different off-step points to obtain multiple hybrid method and presented as a block method. The resulting methods are zero-stable, consistent and convergent. Some numerical examples were given to demonstrate the accuracy and efficiency of the proposed method and found to give better approximation than the existing methods.

***Keywords*: Bernstein polynomial, Collocation, Interpolation, Hybrid, Block method.**

1. Introduction

In this paper, we present the Bernstein polynomial of degree m defined on the interval [a,b], given by [1] and [2] in the following form.

$$y(x) = \sum_{k=0}^{m=r+s-1} c_k \binom{m}{i} \frac{(x-a)^i (b-x)^{m-i}}{(b-a)^m}, \qquad k = 0,1,\dots,m \qquad (1)$$

to find an approximation solution to general second order initial value problem (IVP) of Eq (2)

$$y'' = f\big(x, y(x), y'(x)\big) \;\; y(x_0) = \eta_0, y'(x_0) = \eta_1, x\epsilon[a,b] \qquad (2)$$

where r and s are number of distinct collocation and interpolation points respectively, c_k are the coefficients to be determined and for $m \geq 1$. In most cases researchers reduces equation (2) to systems of first order (IVP) and then, suitable methods for first order equations are adopted to solve them. Reduction approach is quite good but has been known to have some drawbacks which include waste of time, tediousness, the need for large computer storage and alots of human efforts, this approach has been extensively discussed in the literature [3, 4]. Because of this, many researchers have attempted to solve equation (2) directly, among are those of [5-8].

Direct method for solving (2) was implemented in different ways such as predictor-corrector method, block method, which provide starting points for predictor-corrector method and hybrid block method. Hybrid block method combined step and off-step points to form a single block for solving ODEs, which enable to overcome the drawback of block methods, [9, 10].

Various methods have been proposed by many authors for solving (2) using different basis functions. This includes, Alkasassbeh and Omar [11] adopted the power series as basis function, Olabode and Momoh [12] used Chebyshev polynomials. In Jator [9], finite difference method was used. Other basis polynomials used for approximate solution to (2) are Lucas polynomials in Adeniran and Longe [13], shifted roots of Legendre polynomials in Kamoh et al [8] and Ukpebor [14], and Trigonometric polynomials in Adeniran and Longe [7].

In this paper, we developed an order seven two-step hybrid block method to extend the work of Ojo and Okoro [15-16] which derived one-step block method for solving (2) directly without reducing system of first-order using Bernstein polynomial

2. Derivation of the Method

In this section, we shall construct a two-step continuous LMMs. To achieve this, we imposed the following conditions on (1)

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$$\bar{y}\left(x_{n+\frac{j}{6}}\right) = y_{n+\frac{j}{6}}, j = 0, 1, 9 \qquad (3)$$

$$\bar{y}''\left(x_{n+\frac{j}{6}}\right) = f\left(\left(x_{n+\frac{j}{6}}\right), y\left(x_{n+\frac{j}{6}}\right), y'\left(x_{n+\frac{j}{6}}\right)\right), j = 0, 1, 2, \ldots, 6k \qquad (4)$$

where k is the step number.

Now, by interpolating equation (1) at $x = x_{n+\frac{j}{6}}, j = 0, 1, 9$ and collocating its second derivative at all points $x = x_{n+\frac{j}{6}}, j = 0, 1, 2, \ldots, 6k$ respectively, gives a system of non-linear equation of the form

$$QX = B \qquad (5)$$

Solving (5) for the c_k, $k = 0(1)8$ and substituting back into (1) above and after much algebraic simplification yield a method of the form

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h^2 \left[\sum_{j=0}^{k} \beta_j(x) f_{n+j} + \sum_{\nu}^{k} \beta_\nu(x) f_{n+\nu}\right] \qquad (6)$$

where $y(x)$ is the numerical solution of the ivp and $\nu = \frac{1}{6}, \frac{1}{2}, 1, \frac{4}{,3}, \frac{3}{2}, \alpha_j$ and β_j are constants and

$f_{n+j} = f(x_{n+j},\ y_{n+j}, y'_{n+j})$.

Equation (6) is evaluated at the non-interpolating points $x = x_{n+\frac{j}{6}}, j = 0,1,3,6,8,12$ and its first derivative at all points $x = x_{n+\frac{j}{6}}, j = 0, 1, 2, \ldots, 6k$ produces the following general equations in block form

$$\boldsymbol{AY_L = BR_1 + CR_2 + DR_3} \qquad (\mathbf{7})$$

Where

$$\boldsymbol{Y_L} = \begin{bmatrix} y_{n+\frac{1}{6}} \\ y_{n+\frac{1}{2}} \\ y_{n+1} \\ y_{n+\frac{4}{3}} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \\ y'_{n+\frac{1}{6}} \\ y'_{n+\frac{1}{2}} \\ y'_{n+1} \\ y'_{n+\frac{4}{3}} \\ y'_{n+\frac{3}{2}} \\ y'_{n+2} \end{bmatrix}. \qquad \boldsymbol{R_1} = \begin{bmatrix} y_n \\ y'_n \end{bmatrix}, \qquad \boldsymbol{R_2} = [\boldsymbol{f_n}], \qquad \boldsymbol{R_3} = \begin{bmatrix} f_{n+\frac{1}{6}} \\ f_{n+\frac{1}{2}} \\ f_{n+1} \\ f_{n+\frac{4}{3}} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix}, \qquad \boldsymbol{C} = \begin{bmatrix} \frac{58409h^2}{52254720} \\ -\frac{491h^2}{612360} \\ -\frac{36329h^2}{67184640} \\ -\frac{64037h^2}{31352832} \\ -\frac{1002221h^2}{156764160} \\ -\frac{58415h^2}{1119744} \\ \frac{886h^2}{76545} \\ -\frac{32299h^2}{2449440} \\ \frac{245507h^2}{39191040} \\ \frac{115091h^2}{39191040} \\ \frac{274h^2}{76545} \\ -\frac{1546861h^2}{39191040} \end{bmatrix}$$

$$A = \begin{bmatrix} -9/8 & & & & 1/8 & & & & & & & \\ -3/4 & 0 & 0 & 0 & -1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/8 & 1 & 0 & 0 & -7/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3/8 & 0 & 0 & 1 & -5/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3/8 & 0 & 0 & 0 & -11/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3/4 & 0 & 1 & 0 & -3/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3/4 & 0 & 0 & 0 & -3/4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3/4 & 0 & 0 & 0 & -3/4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3/4 & 0 & 0 & 0 & -3/4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 3/4 & 0 & 0 & 0 & -3/4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3/4 & 0 & 0 & 0 & -3/4 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3/4 & 0 & 0 & 0 & -3/4 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 3/4 & 0 & 0 & 0 & -3/4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & h \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$
\boldsymbol{D}=\begin{bmatrix}
\frac{138863h^2}{4139520} & \frac{36101h^2}{653184} & \frac{71159h^2}{2177280} & \frac{101h^2}{1505280} & \frac{31051h^2}{13063680} & -\frac{17023h^2}{287400960}\\
-\frac{1117h^2}{388080} & -\frac{27401h^2}{306180} & -\frac{1838h^2}{25515} & \frac{109h^2}{17640} & -\frac{9617h^2}{1224720} & \frac{727h^2}{3367980}\\
\frac{1643h^2}{1774080} & -\frac{86473h^2}{4199040} & -\frac{133847h^2}{2799360} & -\frac{5821h^2}{215040} & -\frac{36571h^2}{16796160} & \frac{17023h^2}{369515520}\\
\frac{9985h^2}{2483712} & -\frac{125737h^2}{1959552} & -\frac{165491h^2}{1306368} & -\frac{10769h^2}{903168} & -\frac{60955h^2}{7838208} & \frac{4062h^2}{172440576}\\
\frac{22103h^2}{1128960} & \frac{253979h^2}{9797760} & \frac{1521949h^2}{6531840} & -\frac{133573h^2}{903168} & \frac{12547601h^2}{39191040} & \frac{1093457h^2}{78382080}\\
\frac{384089\mathrm{h}}{1552320} & -\frac{391219\mathrm{h}}{1224720} & -\frac{66821\mathrm{h}}{326592} & \frac{9791\mathrm{h}}{1128960} & -\frac{92509\mathrm{h}}{4898880} & \frac{110207\mathrm{h}}{215550720}\\
-\frac{208\mathrm{h}}{1617} & -\frac{26224\mathrm{h}}{76545} & -\frac{688\mathrm{h}}{3645} & -\frac{2\mathrm{h}}{245} & -\frac{158\mathrm{h}}{15309} & \frac{188\mathrm{h}}{841995}\\
\frac{9671\mathrm{h}}{194040} & -\frac{10342\mathrm{h}}{76545} & -\frac{24967\mathrm{h}}{102060} & -\frac{3209\mathrm{h}}{70560} & -\frac{22529\mathrm{h}}{612360} & \frac{2869\mathrm{h}}{2694384}\\
\frac{6431\mathrm{h}}{44352} & -\frac{172541\mathrm{h}}{1224720} & \frac{198551\mathrm{h}}{1632960} & -\frac{20263\mathrm{h}}{161280} & -\frac{38263\mathrm{h}}{979776} & -\frac{212497\mathrm{h}}{215550720}\\
-\frac{2419\mathrm{h}}{517440} & -\frac{149453\mathrm{h}}{1224720} & \frac{68129\mathrm{h}}{233280} & -\frac{36373\mathrm{h}}{376320} & -\frac{43357\mathrm{h}}{4898880} & -\frac{6221\mathrm{h}}{43110144}\\
-\frac{158\mathrm{h}}{24255} & \frac{9584\mathrm{h}}{76545} & \frac{7184\mathrm{h}}{25515} & \frac{442\mathrm{h}}{2205} & \frac{4808\mathrm{h}}{76545} & -\frac{52\mathrm{h}}{120285}\\
\frac{175783\mathrm{h}}{1552320} & -\frac{80179\mathrm{h}}{1224720} & \frac{244219\mathrm{h}}{326592} & -\frac{1109953\mathrm{h}}{1128960} & \frac{884357\mathrm{h}}{699840} & \frac{28041599\mathrm{h}}{215550720}
\end{bmatrix}
$$

D Multipling equation (7) by the inverse of (A) gives the hybrid block method of the form

$$\boldsymbol{IY_L}=\boldsymbol{\bar{B}R_1}+\boldsymbol{\bar{C}R_2}+\boldsymbol{\bar{D}R_3} \qquad (\mathbf{8})$$

Where

$$
I=\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix},\quad
\bar{B}=\begin{bmatrix}
1 & h/6\\
1 & h/2\\
1 & h\\
1 & 4h/3\\
1 & 3h/2\\
1 & 2h\\
0 & 1\\
0 & 1\\
0 & 1\\
0 & 1\\
0 & 1\\
0 & 1
\end{bmatrix},\quad
\bar{C}=\begin{bmatrix}
\frac{3563369h^2}{470292480}\\
\frac{2227h^2}{92160}\\
\frac{247h^2}{5040}\\
\frac{15592h^2}{229635}\\
\frac{5529h^2}{71680}\\
\frac{61h^2}{630}\\
\frac{277573\mathrm{h}}{4354560}\\
\frac{6287\mathrm{h}}{161280}\\
\frac{589\mathrm{h}}{10080}\\
\frac{1406\mathrm{h}}{25515}\\
\frac{999\mathrm{h}}{17920}\\
\frac{4\mathrm{h}}{315}
\end{bmatrix}
$$

$$\bar{\bar{D}} = \begin{bmatrix} \frac{286589h^2}{37255680} & -\frac{59669h^2}{29393280} & \frac{27779h^2}{19595520} & -\frac{20491h^2}{13547520} & \frac{90577h^2}{117573120} & \frac{9601h^2}{369515520} \\ \frac{120447h^2}{1379840} & \frac{67h^2}{4480} & -\frac{13h^2}{5376} & \frac{891h^2}{501760} & -\frac{127h^2}{161280} & \frac{23h^2}{1182720} \\ \frac{2349h^2}{10780} & \frac{h^2}{5} & \frac{19h^2}{420} & -\frac{81h^2}{3920} & \frac{11h^2}{1260} & -\frac{h^2}{4620} \\ \frac{2163h^2}{72765} & \frac{80384h^2}{229635} & \frac{2944h^2}{15309} & -\frac{256h^2}{6615} & \frac{4736h^2}{229635} & -\frac{208h^2}{360855} \\ \frac{465831h^2}{1379840} & \frac{1899h^2}{4480} & \frac{351h^2}{1280} & -\frac{6561h^2}{501760} & \frac{93h^2}{3584} & -\frac{279h^2}{394240} \\ \frac{1296h^2}{2695} & \frac{64h^2}{105} & \frac{64h^2}{105} & -\frac{81h^2}{490} & \frac{16h^2}{45} & \frac{h^2}{77} \\ \frac{184409\text{h}}{1552320} & -\frac{1891\text{h}}{81648} & \frac{8627\text{h}}{544320} & -\frac{19007\text{h}}{1128960} & \frac{4661\text{h}}{544320} & -\frac{20693\text{h}}{71850240} \\ \frac{51273\text{h}}{172480} & \frac{929\text{h}}{5040} & -\frac{269\text{h}}{6720} & \frac{4617\text{h}}{125440} & -\frac{361\text{h}}{20160} & \frac{491\text{h}}{887040} \\ \frac{2511\text{h}}{10780} & \frac{29\text{h}}{63} & \frac{137\text{h}}{420} & -\frac{1053\text{h}}{7840} & \frac{73\text{h}}{1260} & -\frac{83\text{h}}{55440} \\ \frac{5888\text{h}}{24255} & \frac{11264\text{h}}{25515} & \frac{1408\text{h}}{2835} & \frac{194\text{h}}{2205} & \frac{256\text{h}}{25515} & -\frac{184\text{h}}{280665} \\ \frac{41553\text{h}}{172480} & \frac{249\text{h}}{560} & \frac{1089\text{h}}{2240} & \frac{24057\text{h}}{125440} & \frac{183\text{h}}{2240} & -\frac{93\text{h}}{98560} \\ \frac{972\text{h}}{2695} & \frac{16\text{h}}{63} & \frac{20\text{h}}{21} & -\frac{243\text{h}}{245} & \frac{404\text{h}}{315} & \frac{449\text{h}}{3465} \end{bmatrix}$$

Expanding the block method (8) in Taylor Series and comparing the coefficients in h gives

$$\begin{pmatrix}
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{6}\right)^j h^j}{j!}y_n^{(j)} - y_n - \frac{1}{6}hy'_n - \frac{3563369}{470292480}h^2y''_n - \sum_{j=0}^{\infty}\frac{h^{(j+2)}}{j!}y_n^{(j+2)}\left[\frac{286589}{37255680}\left(\frac{1}{6}\right)^j - \frac{59669}{29393280}\left(\frac{1}{2}\right)^j + \frac{27779}{19595520}(1)^j - \frac{20491}{13547520}\left(\frac{4}{3}\right)^j + \frac{90577}{117573120}\left(\frac{3}{2}\right)^j - \frac{9601}{369515520}(2)^j\right] \\
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)^j h^j}{j!}y_n^{(j)} - y_n - \frac{1}{2}hy'_n - \frac{2227}{92160}h^2y''_n - \sum_{j=0}^{\infty}\frac{h^{(j+2)}}{j!}y_n^{(j+2)}\left[\frac{120447}{1379840}\left(\frac{1}{6}\right)^j + \frac{67}{4480}\left(\frac{1}{2}\right)^j - \frac{13}{5376}(1)^j + \frac{891}{501760}\left(\frac{4}{3}\right)^j - \frac{127}{161280}\left(\frac{3}{2}\right)^j - \frac{23}{1182720}(2)^j\right] \\
\sum_{j=0}^{\infty}\frac{(1)^j h^j}{j!}y_n^{(j)} - y_n - hy'_n - \frac{247}{5040}h^2y''_n - \sum_{j=0}^{\infty}\frac{h^{(j+2)}}{j!}y_n^{(j+2)}\left[\frac{2349}{10780}\left(\frac{1}{6}\right)^j + \frac{1}{5}\left(\frac{1}{2}\right)^j + \frac{19}{420}(1)^j - \frac{81}{3920}\left(\frac{4}{3}\right)^j + \frac{11}{1260}\left(\frac{3}{2}\right)^j - \frac{1}{4620}(2)^j\right] \\
\sum_{j=0}^{\infty}\frac{\left(\frac{4}{3}\right)^j h^j}{j!}y_n^{(j)} - y_n - \frac{4}{3}hy'_n - \frac{15592}{229635}h^2y''_n - \sum_{j=0}^{\infty}\frac{h^{(j+2)}}{j!}y_n^{(j+2)}\left[\frac{21632}{72765}\left(\frac{1}{6}\right)^j + \frac{80384}{229635}\left(\frac{1}{2}\right)^j + \frac{2944}{15309}(1)^j - \frac{256}{6615}\left(\frac{4}{3}\right)^j + \frac{4736}{229635}\left(\frac{3}{2}\right)^j - \frac{208}{360855}(2)^j\right] \\
\sum_{j=0}^{\infty}\frac{\left(\frac{3}{2}\right)^j h^j}{j!}y_n^{(j)} - y_n - \frac{3}{2}hy'_n - \frac{5529}{716805}h^2y''_n - \sum_{j=0}^{\infty}\frac{h^{(j+2)}}{j!}y_n^{(j+2)}\left[\frac{465831}{1379840}\left(\frac{1}{6}\right)^j + \frac{1899}{4480}\left(\frac{1}{2}\right)^j + \frac{351}{1280}(1)^j - \frac{6561}{501760}\left(\frac{4}{3}\right)^j + \frac{93}{3584}\left(\frac{3}{2}\right)^j - \frac{279}{394240}(2)^j\right] \\
\sum_{j=0}^{\infty}\frac{(2)^j h^j}{j!}y_n^{(j)} - y_n - 2hy'_n - \frac{61}{630}h^2y''_n - \sum_{j=0}^{\infty}\frac{h^{(j+2)}}{j!}y_n^{(j+2)}\left[\frac{1296}{2695}\left(\frac{1}{6}\right)^j + \frac{64}{105}\left(\frac{1}{2}\right)^j + \frac{64}{105}(1)^j + \frac{81}{490}\left(\frac{4}{3}\right)^j + \frac{16}{45}\left(\frac{3}{2}\right)^j - \frac{1}{77}(2)^j\right] \\
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{6}\right)^j h^j}{j!}y_n^{(j+1)} - y'_n - \frac{277573}{4354560}hy^2{}_n - \sum_{j=0}^{\infty}\frac{h^{(j+1)}}{j!}y_n^{(j+2)}\left[\frac{184409}{1552320}\left(\frac{1}{6}\right)^j - \frac{1891}{81648}\left(\frac{1}{2}\right)^j + \frac{8627}{544320}(1)^j - \frac{19007}{1128960}\left(\frac{4}{3}\right)^j + \frac{4661}{544320}\left(\frac{3}{2}\right)^j - \frac{20693}{7185024}(2)^j\right] \\
\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)^j h^j}{j!}y_n^{(j+1)} - y'_n - \frac{6287}{16128}hy^2{}_n - \sum_{j=0}^{\infty}\frac{h^{(j+1)}}{j!}y_n^{(j+2)}\left[\frac{51273}{172480}\left(\frac{1}{6}\right)^j + \frac{929}{5040}\left(\frac{1}{2}\right)^j - \frac{269}{6720}(1)^j + \frac{4617}{125440}\left(\frac{4}{3}\right)^j - \frac{361}{20160}\left(\frac{3}{2}\right)^j + \frac{491}{887040}(2)^j\right] \\
\sum_{j=0}^{\infty}\frac{(1)^j h^j}{j!}y_n^{(j+1)} - y'_n - \frac{589}{10080}hy^2{}_n - \sum_{j=0}^{\infty}\frac{h^{(j+1)}}{j!}y_n^{(j+2)}\left[\frac{2511}{10780}\left(\frac{1}{6}\right)^j + \frac{29}{63}\left(\frac{1}{2}\right)^j + \frac{137}{420}(1)^j - \frac{1053}{7840}\left(\frac{4}{3}\right)^j + \frac{73}{1260}\left(\frac{3}{2}\right)^j - \frac{83}{55440}(2)^j\right] \\
\sum_{j=0}^{\infty}\frac{\left(\frac{4}{3}\right)^j h^j}{j!}y_n^{(j+1)} - y'_n - \frac{1406}{25515}hy^2{}_n - \sum_{j=0}^{\infty}\frac{h^{(j+1)}}{j!}y_n^{(j+2)}\left[\frac{5888}{24255}\left(\frac{1}{6}\right)^j + \frac{11264}{25515}\left(\frac{1}{2}\right)^j + \frac{1408}{2835}(1)^j + \frac{194}{2205}\left(\frac{4}{3}\right)^j - \frac{256}{25515}\left(\frac{3}{2}\right)^j + \frac{184}{280665}(2)^j\right] \\
\sum_{j=0}^{\infty}\frac{\left(\frac{3}{2}\right)^j h^j}{j!}y_n^{(j+1)} - y'_n + \frac{999}{17920}hy^2{}_n - \sum_{j=0}^{\infty}\frac{h^{(j+1)}}{j!}y_n^{(j+2)}\left[\frac{41553}{172480}\left(\frac{1}{6}\right)^j + \frac{249}{560}\left(\frac{1}{2}\right)^j + \frac{1089}{2240}(1)^j + \frac{24057}{125440}\left(\frac{4}{3}\right)^j + \frac{183}{2240}\left(\frac{3}{2}\right)^j - \frac{93}{98560}(2)^j\right] \\
\sum_{j=0}^{\infty}\frac{(2)^j h^j}{j!}y_n^{(j+1)} - y'_n + \frac{4}{315}hy^2{}_n - \sum_{j=0}^{\infty}\frac{h^{(j+1)}}{j!}y_n^{(j+2)}\left[\frac{972}{2695}\left(\frac{1}{6}\right)^j + \frac{16}{63}\left(\frac{1}{2}\right)^j + \frac{20}{21}(1)^j + \frac{243}{245}\left(\frac{4}{3}\right)^j + \frac{404}{315}\left(\frac{3}{2}\right)^j + \frac{449}{3465}(2)^j\right]
\end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Our calculations revealed that the schemes (8) have uniform order $p = 7$ with error constants

$$c_{p+2} = \left(\frac{586301}{31998700339200}, \frac{-19}{9754214400}, \frac{13}{152409600}, \frac{487}{1953045675}, \frac{19}{60211200}, \frac{-1}{2381400}\right)^T$$

3 Zero Stability and Convergence of the Method

The zero-stability is concerned with the stability of the difference system in the limit as h tends to zero. Thus, as $h \to 0$, the method (8) tends to the difference system.

$$\boldsymbol{IY_L - \bar{B}R_1 = 0} \qquad \textbf{(9)}$$

whose first characteristic polynomial $p(z)$ is given by $p(z) = \det|zI - \boldsymbol{\bar{B}}|$

$$p(z) = det0 \left| z \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} - \begin{bmatrix} 0&0&0&0&0&1&0&0&0&0&0&h/6 \\ 0&0&0&0&0&1&0&0&0&0&0&h/2 \\ 0&0&0&0&0&1&0&0&0&0&0&h \\ 0&0&0&0&0&1&0&0&0&0&0&4h/3 \\ 0&0&0&0&0&1&0&0&0&0&0&3h/2 \\ 0&0&0&0&0&1&0&0&0&0&0&2h \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} \right|$$

$$\rho(z) = z^{10}(z-1)^2 = 0 \qquad (10)$$

Following Fatunla [17] the block method (8) is zero-stable, since from (10), $\rho(z) = 0$ satisfies $|z_i| \le 1, i = 1, \dots, k$ and for those roots with $|z_i| = 1$, the multiplicity must not exceed two. Block method (8) is therefore consistent as it has order p > 1. Hence the convergence of our method is as asserted in Henrici [18]

4. Numerical Experiment and Results

In this section, four problems were considered to test the performance of our method, Bernstein Hybrid Multistep Method (BHMM)

Problem 4.1: We consider the highly stiff IVP which was earlier studied by Jator [9] and also solved by Adeniran and Longe [13]

$$y'' + 1001y' + 1000y = 0, y(0) = 1, y'(0) = -1$$

With the exact solution: $y(x) = e^{-x}$

Table 1: Comparison of the error of our method (BHMM) with a sixth order linear multistep method Jator [9] and Adeniran and Longe [13] for problem 4.1

X	Exact Solution	BHMM Solution	Error in BHMM	Error in [9]	Error in [13]
0.1	0.904837418035959	0.904837418035959	2.5×10^{-19}	6.987×10^{-12}	3.332×10^{-09}
0.2	0.818730753077981	0.818730753077981	4.2×10^{-19}	1.003×10^{-12}	6.388×10^{-09}
0.3	0.740818220681717	0.740818220681717	5.7×10^{-19}	7.859×10^{-12}	9.158×10^{-09}
0.4	0.670320046035639	0.670320046035639	6.9×10^{-19}	10.478×10^{-12}	1.164×10^{-08}
0.5	0.606530659712633	0.606530659712633	7.8×10^{-19}	63.221×10^{-12}	1.383×10^{-08}
0.6	0.548811636094026	0.548811636094026	8.4×10^{-19}	10.050×10^{-12}	1.575×10^{-08}
0.7	0.496585303791409	0.496585303791409	8.9×10^{-19}	9.363×10^{-12}	1.740×10^{-08}
0.8	0.449328964117221	0.449328964117221	9.2×10^{-19}	2.647×10^{-12}	1.880×10^{-08}
0.9	0.406569659740599	0.406569659740599	9.4×10^{-19}	10.679×10^{-12}	1.995×10^{-08}
1.0	0.367879441171442	0.367879441171442	0.0	23.273×10^{-12}	2.088×10^{-08}

Problem 4.2: In this example, we test the performance of our method (BHMM) on the mildly stiff problem which was also solved by Alkasassbeh and Omar [11] and Ukpebor [14].
$f(x, y, y') = y', y(0) = 0, y'(0) = -1$
With the exact solution: $y(x) = (1 - e^x)$
It is observed that our method performs better than those given in Alkasassbeh and Omar [11] and Ukpebor [14] despite the fact that we used a lager step-size h = 0.01. Hence, for this example, our method is clearly superior. As shown in Table 2.

Table 2: Comparison of the error of our method (BHMM) with Alkasassbeh & Omar [11] and Ukpebor [14] for problem 4.2

X	Exact Solution	BHMM Solution	Error in BHMM h=0.01	Error in [11] h=0.01	Error in [14] =0.1/32
0.1	-0.1051709180756476248	-0.10517091807564762484	$4.\times 10^{-20}$	8.33×10^{-17}	9.93×10^{-17}
0.2	-0.2214027581601698339	-0.22140275816016983404	1.4×10^{-19}	2.78×10^{-16}	6.42×10^{-17}
0.3	-0.3498588075760031040	-0.34985880757600310430	3.0×10^{-19}	5.55×10^{-16}	8.71×10^{-16}
0.4	-0.4918246976412703178	-0.49182469764127031842	6.2×10^{-19}	9.44×10^{-16}	5.35×10^{-15}
0.5	-0.6487212707001281468	-0.64872127070012814787	1.07×10^{-18}	2.11×10^{-15}	3.20×10^{-12}
0.6	-0.8221188003905089749	-0.82211880039050897645	1.55×10^{-18}	3.22×10^{-15}	6.40×10^{-12}
0.7	-1.0137527074704765216	-1.0137527074704765239	2.3×10^{-18}	4.44×10^{-15}	9.62×10^{-12}
0.8	-1.2255409284924676046	-1.2255409284924676077	3.1×10^{-18}	5.99×10^{-15}	1.29×10^{-11}
0.9	-1.4596031111569496638	-1.4596031111569496680	4.2×10^{-18}	7.77×10^{-15}	3.21×10^{-11}
1.0	-1.7182818284590452354	-1.7182818284590452410	5.6×10^{-18}	1.07×10^{-14}	5.15×10^{-11}

Problem 4.3: Here we consider the nearly periodic Stiefel and Bettis initial value problem, which was also studied in [9], [12] and [19],

$$y'' + y = \frac{1}{1000}e^{ix}, y(0) = 1, y'(0) = 0.9995i, x \in [0, \pi]$$

which has the equivalent form problems

$$y''_1 + y_1 = \frac{1}{1000}\cos(x), y_1(0) = 1, y'_1(0) = 0$$

$$y''_2 + y_2 = \frac{1}{1000}\sin(x), y_2(0) = 1, y'_2(0) = 0.9995$$

$$y(x) = y_1(x) + iy_2(x);\ y_1, y_2 \in \mathcal{R}, \mathcal{D}(x) = \sqrt{y^2{}_1(x) + y^2{}_2(x)}$$

with the following theoretical solution: $y_1(x) = \cos(x) + \frac{1}{2000}x\sin(x)$, and $y_2(x) = \sin(x) - \frac{1}{2000}x\cos(x)$. According to Olabode and Momoh [12], the differential system in the above problem, represents motion on a perturbed circular orbit in complex plane in which the point $y(x)$ spirals slowly outward such that its distance from the origin at any given time t is $\mathcal{D}(x)$. With the exact solution of $\mathcal{D}(x) = 1.001972$.

Table 3: Comparison of the error of our method (BHMM) with Chebyshev Hybrid Multistep Method Olabode & Momoh [12] for problem 4.3,

X	Exact Solution	(y_1) BHMM Solution	Error in (y_1) BHMM	Error in [12]
0.0062500	0.999980488345	0.999980488345	0.0000E + 000	4.98E − 18
0.0093750	0.999956098954	0.999956098954	0.0000E + 000	3.34E − 18
0.0125000	0.999921954140	0.999921954140	1.1102E − 016	9.96E − 18
0.0156250	0.999878054236	0.999878054236	0.0000E + 000	1.65E − 18
0.0187500	0.999824399671	0.999824399671	0.0000E + 000	1.49E − 17
0.0218750	0.999760990967	0.999760990967	0.0000E + 000	6.63E − 18
0.0250000	0.999687828743	0.999687828743	0.0000E + 000	1.99E − 17
0.0281250	0.999604913714	0.999604913714	0.0000E + 000	1.16E − 17
0.0312500	0.999512246687	0.999512246687	1.1102E − 016	2.49E − 17

X	Exact Solution	(y_2) BHMM Solution	Error in (y_2) BHMM	Error in [12]
0.0062500	0.006246834371	0.006246834371	0.0000E + 000	1.90E − 20
0.0093750	0.009370175377	0.009370175377	0.0000E + 000	2.43E − 20
0.0125000	0.012493424970	0.012493424970	5.2042E − 018	1.10E − 19
0.0156250	0.015616552679	0.015616552679	6.9389E − 018	2.10E − 20
0.0187500	0.018739528034	0.018739528034	0.0000E + 000	2.44E − 19
0.0218750	0.021862320570	0.021862320570	3.4694E − 018	1.27E − 19
0.0250000	0.024984899821	0.024984899821	0.0000E + 000	4.49E − 19
0.0281250	0.028107235322	0.028107235322	6.9389E − 018	2.96E − 19
0.0312500	0.031229296614	0.031229296614	0.0000E + 000	7.16E − 19

Table 4, shown the solution for problem 4.3 obtain the values of $y_1(x)$ and $y_2(x)$ using the our method (BHMM) for $\pi/4$, $\pi/5, \pi/6, \pi/9, \pi/12$

Table 4: Comparison of the error of our method (BHMM) with Jator [9], Olabode & Momoh [12], Lambert & Watson [19], for problem 4.3,

h	[9]	[12]	[19]	Error in our method (BHMM)
$\pi/4$	1.003067	1.002084	1.003145	1.0021842
$\pi/5$	1.002217	1.002117	1.002312	1.0021172
$\pi/6$	1.002047	1.002064	1.002048	1.0020545
$\pi/9$	1.001978	1.001984	1.001982	1.0019826
$\pi/12$	1.001973	1.001974	1.001971	1.0019719

Problem 4.4: We consider the non-linear IVP which was also solved by Adesanya & Odekunle [20] and Anake, et al [21] for the step-size h = 0.05.

$y'' - x(y')^2 = 0, y(0) = 1, y'(0) = 1/2$

With the exact solution: $y(x) = 1 + \frac{1}{2} In((2 + x)/(2 - x))$

Table 5: Comparison of the error of our method (BHMM) with Adesanya & Odekunle [20] and Anake, et al [21] for problem 4.4

X	Exact Solution	BHMM Solution	Error in BHMM	Error in [20]	Error in [21]
0.1	1.050041729278	1.050041729278	9.1038E − 15	7.5028E − 13	2.5056E – 12
0.2	1.100335347731	1.100335347731	2.7400E − 13	9.7410E − 12	2.0446E − 11
0.3	1.151140435936	1.151140435934	2.1492E − 12	3.7638E − 11	7.0966E − 11
0.4	1.202732554054	1.202732554045	9.4194E − 12	9.7765E − 11	1.7482E − 10
0.5	1.255412811883	1.255412811853	3.0318E − 11	2.0825E − 10	3.5904E − 10
0.6	1.309519604203	1.309519604123	3.0434E − 11	3.9604E − 10	6.6068E − 10
0.7	1.365443754271	1.365443754083	1.8795E − 10	7.0460E − 10	1.1328E − 09
0.8	1.423648930194	1.423648929791	1.0260E − 10	1.2095E − 09	1.8543E − 09
0.9	1.484700278594	1.484700277783	2.1127E − 10	2.0511E − 09	2.9461E − 09
1.0	1.549306144334	1.549306142766	1.5684E − 09	3.5066E − 09	4.6013E − 09

5 Conclusion

A two-step Bernstein Hybrid Multistep Method (BHMM) of order 7 for the direct solution of general second order ODEs was proposed. The block method is derived through the technique of interpolation and collocation at appropriate selected points. The stability properties of the developed method compete favourably with the existing method as shown in the numerical examples.

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